Framing and Ambiguity*

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Abstract

In this paper, we develop and analyze a model of framing under ambiguity. Frames are circumstances, unobservable to the analyst, that shape the agent's perception of the relevant ambiguity. The analyst observes a choice correspondence that represents the set of possible choices under the various decision frames. Our first result provides axioms that are equivalent to a multi-multiple prior model; that is, there is a collection of multiple prior models with a common utility index so that the choice correspondence consists of the optimizers of the models in the collection. Furthermore, we characterize the extent to which the analyst can identify the parameters of the model, that is, the extent to which the frames can be inferred from behavior. To capture the degree to which frames affect choice, we introduce two comparative notions; the first says that one agent is *more decisive* than another if the former's choice correspondence is a subset of the latter's. The second, less demanding notion says that an agent is *more consistent* than another agent if the former has a unique choice whenever the latter does. We characterize both comparative measures in terms of the model parameters. Agents who recognize that they are subject to different frames may learn by combining their frames into a single model. Our final result characterizes the behavioral implications of this form of learning.

Keywords: Framing Effects, Ambiguity, Multiple Priors, Aggregation of Opinions

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1 Introduction

In this paper, we present a model of decision making under uncertainty with framing effects. More specifically, we analyze decision-makers who confront Knightian uncertainty and whose reasoning about this uncertainty is influenced by the manner in which the choice problem is presented. Each frame suggests a different assessment of the underlying uncertainty and therefore, may yield a different choice from a given set of options. A modeler who cannot observe the frame cannot rule out any one of these choices. Once the frame is fixed, the decision maker is a maxmin expected utility maximizer. The frame determines the set of probabilities over which expected utility is minimized.

Our motivation is the observation that decision makers are constrained by limited cognitive resources; they suffer from limited attention, limited memory or computational ability or a coarse understanding of contingencies. Such decision makers cannot identify, assess and integrate all available payoff-relevant information into unified decision procedure. In the absence of such a coherent procedure, the decision maker may be prone to mistakes and biases and may end up making different choices in seemingly identical situations.

To illustrate our framework, we consider a hypothetical student, Alice, preparing for an exam of unknown difficulty. She needs to decide whether to spend the day studying for the exam or work on her history paper. She mentions the exam to one of her friends who asks whether Alice thinks the exam will be *hard* or *not hard*. Alice realizes that if the exam is hard, she is likely to fail no matter how she spends the day. However, if the exam is not hard, an extra day of studying is likely to make the difference between success and failure. Given a choice between failing for sure and failing if the exam is hard, Alice decides to devote her day to studying for the exam.

Consider the following alternative scenario in which Alice, instead of discussing the exam with a friend, mentions it to her brother who states that in his opinion, the key is to assess whether the exam will be *easy* or *not easy*. Alice guesses that if the exam is *not easy*, then there is a possibility of a failure whether she studies for the exam or not, and if the exam is *easy*, she will likely pass even if she spends the day on her history paper. Based on this reasoning, Alice concludes that she should devote her day to the history paper.

An outside observer studying Alice's behavior will know whether she spent day studying for the exam or working on her history paper and whether she passed the exam or not and might even learn whether the exam was easy, hard or neither (that is, in-between).

However, the observer is unlikely to know whether Alice's framing of the problem was influenced by her friend or her sister. Hence, to the outside observer, Alice can take either action, study for the exam or work on her paper.

Notice that each frame yields an incomplete specification of the underlying uncertainty. We interpret this incompleteness as ambiguity. With the friend's framing, Alice contemplates the events *hard* and *not hard* and hence these events become unambiguous while the events *easy* and *neither easy nor hard* are suppressed and remain ambiguous. With the brother's framing, Alice contemplates the events *easy* and *not easy* and hence these events become unambiguous while the events *hard* and *neither easy nor hard* are suppressed and remain ambiguous.

The primitive of our *Framed Ambiguity* model is a choice correspondence which maps every non-empty compact set of Anscombe-Aumann acts to a subset of possible choices from that set. Our axioms on this correspondence yield the following representation:

$$\begin{aligned} c(A) &= \bigcup_i c_i(A) \\ c_i(A) &= \left\{ f \in A \;\middle|\; \min_{\mu \in P_i} E_\mu U(f) \geq \min_{\mu \in P_i} E_\mu U(g) \;\forall g \in A \right\} \end{aligned}$$

for all i. Hence, each frame i identifies a set of probability distributions P_i describing the uncertainty associated with that frame, while vNM expected utility function U remains the same for all frames. After adopting a particular frame, the agent becomes a maxmin expected utility maximizer (Gilboa and Schmeidler (1989)).

The three key axioms of our representation theorem (Theorem 1) are *Framed Uncertainty*, a weakening of WARP that retains Sen's α (sometimes called contraction consistency) but replaces Sen's β (expansion consistency) with a weaker alternative, a new axiom, *No-C-Hedging*, which extends Gilboa and Schmeidler's c-independence for decision makers who fail WARP, and *Indirect Ambiguity Aversion*, which is a version of decision maker's aversion to ambiguity, suitable for the environment with framing. Hence, *framed ambiguity* is a multi-multiple prior model that has maxmin expected utility as a special case; when there is a unique frame. This is the only case in which the frame ambiguity model satisfies *WARP*.

With multiple frames, in general, it is not possible to identify uniquely the parameters of the representation. In Theorem 2, we show that decision maker's expected utility function is unique up to a positive affine transformation and that there is a *minimum* and a *maximum* family of frames. The set of all representations has a lattice structure: a family of frames represents the choice function if and only if it includes the minimum family and is included in the maximum family. We provide a full characterization of all representations in Proposition 3. If all sets of beliefs in a representation are singletons; that is, if the decision maker is a subjective expected utility maximizer with frame-dependent beliefs, then the representation is unique (Corollary 1).

We consider two criteria for comparing framed-ambiguity averse agents: decision maker 1 is *more decisive than* decision maker 2, if from each choice set, decision maker 1 chooses a subset of the options that decision maker 2 chooses. Decision maker 1 is *more consistent than* decision maker 2, if whenever decision maker 2 chooses a unique option from a choice set, so does decision maker 1.

In Proposition 5, we show that one decision maker is more decisive than another if and only if the two have the same expected utility functions and the former's maximal family of frames is a subset of the latter's maximal family of frames. Hence, a more decisive agent is one who has fewer frames. Theorem 3 provides a characterization of the more consistent than relationship for decision makers with finitely many frames. Unlike greater decisiveness, greater consistency does not require fewer frames; rather, it imposes restrictions on the type of additional frames that a more consistent agent can have.

1.1 Related Literature

Our paper contributes to the literature on framing, on ambiguity and on aggregation of beliefs. In this section, we briefly introduce the related papers. We provide more details in subsequent sections to discuss relation to our results.

Tversky and Kahneman (1981) introduced the notion of framing. In their formulation, framing introduces a benchmark that enables decision makers to identify some outcomes as *gains* and others as *losses*. In our formulation, a frame enables a decision maker to organize her reasoning about the uncertainty.

Framed ambiguity model extends Gilboa and Schmeidler (1989) maxmin model of ambiguity aversion. We enrich it by assuming multiple sets of beliefs and interpreting each set of beliefs as a frame in the spirit of Salant and Rubinstein (2008)[1].

The Anscombe-Aumann framework in Heller (2012) is a special case of our model under two restrictions: all sets of beliefs are singletons, and the family of frames (singleton

[1] We also assume that the analyst does not observe a frame, but observes only a resulting choice correspondence, as in section 3 of Salant and Rubinstein (2008).

beliefs) is convex. The same is true for Lehrer and Teper (2011) with the nuance that they consider preference relation as a primitive.

Frick, Iijima, and Le Yaouanq (2019)—and, independently, Chandrasekher (2019)[2]—also consider a decision maker whose perception of uncertainty is given by a family of sets of beliefs. They assume that decision maker considers all sets of beliefs in the family, and chooses the most optimistic set of beliefs and the most pessimistic belief in a set to evaluate a given act. Thus, their agent respects *WARP* in contrast to ours.

Kopylov (2019) also builds a model where agent may use maxmin models with different sets of beliefs to evaluate Anscombe-Aumann acts. In the language of our model, he assumes that frames are menu-dependent; moreover, each menu corresponds to exactly one frame. We, in contrast, assume that decision maker can use all frames from the family to make choices from each menu. Thus, the two models are not nested.

Lu (2014) studies a random maxmin model. His primitive is a stochastic choice from menus consisting of ex-ante lotteries over Anscombe-Aumann acts. In terms of his model, we assume much less choice information: only the support of the distribution of choices from menus consisting of degenerate lotteries over acts[3] is known.

Other papers that model framing of uncertainty include Bourgeois-Gironde and Giraud (2009), Ahn and Ergin (2010) and Caplin and Martin (2020). Outside of the uncertainty framework, preference identification in the environment with framing is studied, for instance, in Goldin and Reck (2020).

Other decision-theoretic papers that consider violations of rationality in decision making under uncertainty include Ok, Ortoleva, and Riella (2012), Galaabaatar and Karni (2013) and Hara, Ok, and Riella (2019).

Our comparative statics result contributes to the literature on aggregation of preferences under uncertainty. In particular, in Corollary 3 we consider aggregation of opinions of ambiguity averse experts that agree on utilities, but disagree on sets of beliefs. We discuss relation of our results to Crès, Gilboa, and Vieille (2011) and Hill (2011) in Section 5. Other related papers are Nascimento (2012) and Danan, Gajdos, Hill, and Tallon (2016).

[2]To the best of our knowledge, these two papers will be combined.

[3]To connect his (random choice) and our (choice correspondence) primitives, assume that an act is chosen from the menu if the corresponding degenerate lottery over acts is chosen from the menu of the corresponding degenerate lotteries with positive probability—i.e. we consider the *upper choice correspondence, induced by the stochastic choice* from Ok and Tserenjigmid (2019).

1.2 Structure of the Paper

The rest of the paper is organized as follows. Section 2 introduces our setup, axioms and framed ambiguity model and provides characterization and identification theorems. Section 3 elaborates on identification results, Section 4 studies behavioral implications of agent's sensitivity to framing. Section 5 provides our comparative statics results and application to the social choice theory. All proofs are given in the Appendix.

2 Model

We consider an Anscombe-Aumann setup with a finite set of prizes X and a finite set of states of the world S with $|S| \geq 2$. An act $f: S \to \triangle X$ is a mapping from the state space to the set of probabilities (objective lotteries) on X. Denote the set of acts by H with typical elements $f, g, h, \ldots$ and endow it with metric d consistent with a product topology induced by the topology of the point-wise convergence of probabilities on X. The set of acts H is a mixture space with mixture operation defined via $(\lambda f + (1-\lambda)g)(s) = \lambda f(s) + (1-\lambda)g(s)$. A constant act is an act that gives the same consequences in all states: $f(s) = p \ \forall s \in S$. We denote such acts by corresponding lotteries $p, q, r, \ldots$ whenever it does not cause confusion. The set of constant acts is $H_0 \subset H$. Decision maker (DM, she) chooses acts from menus $A, B, C, \ldots$. A menu is a non-empty compact subset of H. We denote the set of all menus by $\mathcal{K}$ and endow it with the Hausdorff metric. A mixture between a menu and an act is defined via $\lambda A + (1-\lambda)g = \{\lambda f + (1-\lambda)g \mid f \in A\}$. The set of menus consisting of constant acts is $\mathcal{K}_0 \subset \mathcal{K}$.

2.1 Choice Correspondence

The analyst observes a *choice correspondence* $c: \mathcal{K} \to 2^H$ such that $\varnothing \neq c(B) \subseteq B$ for any $B \in \mathcal{K}$. We interpret $c(B)$ as a subset of acts that *could be chosen* from menu B by the DM. We make the following implicit assumptions:

(i) DM's perception of uncertainty could be influenced by the details of the decision problem unobserved by the analyst that do not provide DM with additional information about the state of the world. We call the collection of such details *a frame*.

(ii) An analyst is able to recover choice correspondence from the data by observing repeated choices from each menu. Inconsistency of choices emerges only either from varying framing of the problem or from indifferences between the alternatives.

(iii) The collection of frames that DM experiences is invariant across menus.

The first assumption is the premise of the project, as we are interested in the class of choice correspondences that emerge as a result of the *ambiguity framing*. Assumption (ii) in particular says that *no learning* occurs: notwithstanding the fact that DM faces multiple frames for a menu, she is unable to connect her past decisions to the current situation. Assumption (iii) says that we don't consider menu-dependent frames. Finally, we make an assumption about the DM's attitude toward uncertainty:

(iv) Decision maker is cautious toward the uncertainty that she could realize.

Assumption (iv) is common for models involving Knightian uncertainty. Moreover, if DM realizes that she is sensitive to framing, this gives her an additional novel motivation to use caution. DM might think that the uncertainty relevant for the choice problem is framed in a way that erroneously inflates values of some acts—such as emphasizing possible priors that favor these acts. In this case the worst case scenario evaluation could be a good approach to counteract this type of exogenous framing.

2.2 Axioms

We introduce behavioral axioms on the choice correspondence in the spirit of the discussed above implicit assumptions (i)-(iv). Recall that *Weak Axiom of Revealed Preference (WARP)* is equivalent to conditions α and β taken together:

WARP: $\forall A, B \in \mathcal{K} \quad c(A) \cap B \neq \varnothing \implies c(B) \cap A \subseteq c(A)$

$\Updownarrow$

α: $\forall A, B \in \mathcal{K} \quad c(A \cup B) \cap A \subseteq c(A)$

β: $\forall A, B \in \mathcal{K} \quad c(A \cup B) \cap A \neq \varnothing \implies c(A) \subseteq c(A \cup B)$

Our first axiom relaxes *WARP* on the domain of menus that involve non-constant acts.

AXIOM 1 *(Framed Uncertainty):*

1.1 *(α)*: $\forall A, B \in \mathcal{K} \quad c(A \cup B) \cap A \subseteq c(A)$

1.2 *(C-β)*: $\forall A, B \in \mathcal{K}_0 \quad c(A \cup B) \cap A \neq \varnothing \implies c(A) \subseteq c(A \cup B)$

1.3 *(Aizerman's Property)*: $\forall A \in \mathcal{K}\ \ \forall f \in H \quad f \notin c(A \cup \{f\}) \implies c(A) \subseteq c(A \cup \{f\})$

Condition α is appealing, since if there is some framing that allows DM to choose an act from a menu, the same framing would allow DM to chose this act from the subset of this menu. Since we assume that framing influences only the perception of uncertainty, we demand *WARP* to hold on the domain of menus, consisting of constant acts, hence *C-β*. Finally, *Aizerman's Property* relaxes β: one can see that if $f \notin c(A \cup \{f\})$, then $c(A \cup B) \cap A \neq \emptyset$ for $B = \{f\}$, and, hence, $c(A) \subseteq c(A \cup B) = c(A \cup \{f\})$ by β. *Aizerman's Property* says that if h is chosen from the menu, and adding f to this menu results in h not being chosen, then at least f should be good enough by itself to be chosen under some framing.

Our next axiom is a generalization of Gilboa and Schmeidler's c-independence axiom for preference relation to the choice setting:

AXIOM 2 *(C-Independence)*: $\forall A \in \mathcal{K}\ \forall p \in H_0\ \forall \lambda \in (0,1)$

$$c(\lambda A + (1-\lambda)p) = \lambda c(A) + (1-\lambda)p$$

C-Independence says that uncertainty regarding $s \in S$ does not matter for evaluation of constant acts, and that DM is able to factorize Knightian uncertainty and objective uncertainty regarding the prize that lottery $f(s)$ gives in state s. Thus, even though DM may have different perceptions of uncertainty under different frames, her choices do not change once we scale payoffs by varying objective probabilities. Once β is relaxed, though, *C-Independence* does not fully capture this intuition, and we also need the following property:

AXIOM 3 *(No C-Hedging)*: $\forall A \in \mathcal{K}\ \forall f \in H\ \forall p \in H_0\ \forall \lambda \in (0,1)$

$$f, p \in A \implies c(A) \subseteq c(A \cup \{\lambda f + (1-\lambda)p\})$$

To get the intuition behind *No C-Hedging*, suppose that under any circumstances act $\lambda f + (1-\lambda)p$ seems to be so attractive that DM cannot choose h from A in the presence of this act. Disregarding DM's perception of uncertainty either f or p should be at least as attractive as $\lambda f + (1-\lambda)p$, since constant act p cannot effectively hedge against bad outcomes in f. Hence, the presence of both f and p in A would not allow DM to choose h from A under any circumstances as well.

Monotonicity is a natural assumption for the choice under uncertainty. Our version involves only choices from binary menus:

AXIOM 4 *(Strict Monotonicity):* $\forall f, g \in H$
$$g(s) \notin c(\{f(s), g(s)\})\ \forall s \in S \implies g \notin c(\{f, g\})$$

Strict Monotonicity says that DM chooses consequentially. If she cannot find a reason to choose $g(s)$ versus $f(s)$ for any possible contingency, then she concludes that there is no reason to choose g versus f whatever uncertainty regarding $s \in S$ she thinks is possible.

Our last structural axiom formalizes assumption (iv):

AXIOM 5 *(Indirect Ambiguity Aversion):* $\forall A \in \mathcal{K}\ \ \forall h \in A\ \ \forall f, g \in H\ \ \forall \lambda \in (0,1)$
$$h \notin c(A \cup \{f\}) \text{ and } h \notin c(A \cup \{g\}) \implies h \notin c(A \cup \{\lambda f + (1-\lambda) g\})$$

If DM is averse to the ambiguity within the frame, she always considers $\lambda f + (1-\lambda) g$ to be as good as at least one of acts f or g. If adding either f or g to the menu does not allow h to be chosen under any frame, so should do their mixture. Note that the *Indirect Ambiguity Aversion* axiom keeps track of an act that cannot be chosen, and not of the act that is chosen.

Finally, we introduce our technical axioms. We endow $\mathcal{K} \times H$ with a product topology.

AXIOM 6 *(Continuity):* $\{ (A, f) \in \mathcal{K} \times H \mid f \in c(A) \}$ is closed

AXIOM 7 *(C-Non-Degeneracy):* $\exists p, q \in H_0$ such that $\{p\} = c(\{p, q\})$

We use *C-Non-Degeneracy* axiom instead of a weaker statement $c(A) \neq A$ for some $A \in \mathcal{K}$ because if choices over constant acts are degenerate, *Strict Monotonicity* has no bite, and pathological models are consistent with axioms 1–6 and $c(A) \neq A$ for some A[4].

2.3 Representation

We denote by Π the set of all non-empty compact and convex subsets of beliefs $P \subseteq \triangle S$. We endow Π with a Hausdorff metric. Note that Π is compact. When we talk about a set of beliefs P, we assume that $P \in \Pi$.

[4]Lemma 27 in Appendix B.

DEFINITION 1. A *Framed Ambiguity* model is a pair $(U, \mathcal{A})$, where $U: \triangle X \to R$ is a non-degenerate vNM expected utility function, and $\mathcal{A} \subseteq \Pi$ is a non-empty closed family of non-empty compact and convex sets of beliefs $P \in \Pi$.

DEFINITION 2. A framed ambiguity model $(U, \mathcal{A})$ *represents* choice correspondence $c(\cdot)$ if

$$c(B) = \bigcup_{P \in \mathcal{A}} \arg\max_{f \in B} W_P(f) \qquad \forall B \in \mathcal{K} \tag{1}$$

where

$$W_P(f) = \min_{\mu \in P} \sum_{s \in S} \mu(s) U(f(s)) \tag{2}$$

Note that each framed ambiguity model represents a unique choice correspondence given by eq. (1), since $c(B) \subseteq B$, and taking some $P \in \mathcal{A} \neq \varnothing$ we can see that *arg max* is always non-empty. $W_P(\cdot)$ is a utility function of the maxmin model with vNM expected utility $U(\cdot)$ and set of beliefs P. An act belongs to $c(B)$ whenever it maximizes menu B with respect to at least one of utility functions $\{W_P\}_{P \in \mathcal{A}}$. Thus, a framed ambiguity model operationalizes frames via sets of beliefs $P \in \Pi$. This way $\mathcal{A}$ is a set of frames, and each of them could be used as a rational for choosing $f \in c(B)$. We maintain an assumption that a frame is an unobserved property of a decision problem and not a property of an individual act[5]. Thus, if an act is chosen, it should be the best under some framing, when all other acts in the menu are evaluated under the same framing.

Our first main result is a characterization theorem.

Theorem 1. *A choice correspondence has a framed ambiguity representation if and only if Axioms 1–7 hold.*

Theorem 1 says that framed ambiguity model is equivalent to axioms *Framed Uncertainty*, *C-Independence*, *No-C-Hedging*, *Strict Monotonicity*, *Indirect Ambiguity Aversion*, *Continuity* and *C-Non-Degeneracy*. Our model does not contain redundant axioms:

Proposition 1. *Axioms 1–7 are independent.*

[5]To connect the two notions, we can consider a set of frames with a product structure, where each coordinate corresponds to a framing of some act. However, if we want to maintain assumption (iii) about the invariance of the set of frames across menus, we then should consider, for instance, choice from menu $B = \{f, g\}$ with all possible framing of not only acts f and g, but all other acts as well.

Our second main result is identification of the parameters of the model that provides a minimum representation.

Theorem 2. *Let choice correspondence $c(\cdot)$ satisfy axioms 1–7. Then there is a unique minimum family of frames $\mathcal{A} \subseteq \Pi$, a unique maximum family of frames $\mathcal{B} \subseteq \Pi$ and vNM expected utility function U such that $(U', \mathcal{A}')$ represents $c(\cdot)$ if and only if U' is a positive affine transformation of U, and $\mathcal{A} \subseteq \mathcal{A}' \subseteq \mathcal{B}$.*

Theorem 2 tells us that, although we cannot exclude some multiplicity of the representation, there is a well-defined unique minimum collection of frames that provides a framed ambiguity representation of choice correspondence $c(\cdot)$ that satisfies our axioms. The main idea behind this result is that unobserved framing of choice problems induces observed menu effects for the resulting choice correspondence.

To identify frames, for any given act f we find all menus A maximal in the sense that f is chosen from A, but not chosen from any larger menu. If DM satisfies *WARP*, there exists exactly one such menu for any act f—menu, consisting of all acts that are weakly worse than f. In our model, each frame from the minimum family gives rise to a maximum menu A for some act f; this menu consists of all acts that are weakly worse than f under the considered frame.

Theorem 2 also tells us that there exists a maximum representation that includes all possible frames *redundant* for the representation: once there is a minimum collection $\mathcal{A}$, adding or deleting frames in $\mathcal{B} \backslash \mathcal{A}$ does not change the induced choice correspondence. Proposition 3 in the next section characterizes frames in $\mathcal{B}$.

3 Redundant Frames

Consider the following simple example of multiplicity of representation in our model. Let $X = \{x, y\}$ and $S = \{1, 2\}$. Agents 1 and 2 have the same vNM expected utility function $U(p) = p(x)$ for $p \in \triangle X$. The first agent operates with two non-nested intersecting sets of beliefs: $\mathcal{A}_1 = \{P_1, P_2\}$. The second agent uses the same sets of beliefs and also their intersection: $\mathcal{A}_2 = \{P_1, P_2, P_1 \cap P_2\}$. Clearly, $W_{P_1 \cap P_2}(f) = \max\{W_{P_1}(f), W_{P_2}(f)\}$. Thus, if act f maximizes $W_{P_1 \cap P_2}(\cdot)$ in menu B, it also maximizes at least one of $W_{P_1}(\cdot)$ or $W_{P_2}(\cdot)$. It follows that both agents have the same observed choice correspondence. Thus, frame $P_1 \cap P_2$ is redundant for the representation.

DEFINITION 3. Given a non-degenerate vNM expected utility function U and a collection of frames $\mathcal{A} \subseteq \Pi$, frame $P \in \Pi$ is *redundant* if model $(U, \mathcal{A} \cup \{P\})$ represents the same choice correspondence as $(U, \mathcal{A} \backslash \{P\})$.

As we can see from the example, an intersection of frames could be a redundant frame. To extend the idea for larger state spaces, we introduce a notion of *coherent intersection* of compact convex sets (sets of beliefs).

DEFINITION 4. Given a non-empty collection $\mathcal{C}$ of non-empty convex compact sets in $\triangle S \subset R^{|S|}$, a non-empty convex compact set P is a *coherent intersection* of $\mathcal{C}$ if the projection of P on any linear subspace of $R^{|S|}$ is equal to the intersection of projections of sets in the collection $\mathcal{C}$. We say that a coherent intersection is *non-trivial* if $P \notin \mathcal{C}$.

Clearly, if P is a coherent intersection of $\mathcal{C}$, then it is an intersection of frames in $\mathcal{C}$—to see this, take $R^{|S|}$ as a linear subspace of itself. When $\mathcal{C}$ has a coherent intersection P, we say that sets of beliefs in $\mathcal{C}$ intersect coherently. Notice also that any $P \in \Pi$ is a trivial coherent intersection of the collection consisting of P (i.e. $\mathcal{C} = \{P\}$). Recall that a support functional of a set of beliefs P is a function $I_P : R^{|S|} \to R$ given by $I_P(z) = \min\limits_{\mu \in P} \sum\limits_{s \in S} \mu(s) \cdot z_s$. The next proposition shows some properties of the coherent intersections.

Proposition 2. *(i) An intersection P of $\mathcal{C}$ is coherent if and only if $I_P(z) = \max_{P' \in \mathcal{C}} I_{P'}(z)$ for all $z \in R^{|S|}$. (ii) If $|S| = 2$, any non-empty intersection of a non-empty closed collection $\mathcal{C} \subseteq \Pi$ is a coherent intersection of $\mathcal{C}$. (iii) If $|S| > 2$, there is a non-empty closed collection $\mathcal{C} \subseteq \Pi$ that has a non-empty intersection, but does not have a coherent intersection.*

Thus, Proposition 2 gives an alternative definition of the coherent intersection: P is a coherent intersection of $\mathcal{C}$ is when the support functional of P—which is a utility function in the corresponding maxmin model—is equal to the maximum of the support functionals of the sets of beliefs in $\mathcal{C}$. In the proofs, we use this alternative definition.

Let us introduce one more piece of notations.

DEFINITION 5. Let $\mathcal{A} \subseteq \Pi$ be a non-empty collection of sets of beliefs, then its *coherent closure* is given by

$$(\mathcal{A})_{coh} \quad = \quad \{\, P \in \Pi \mid P \textit{ is a coherent intersection of some } \mathcal{C} \subseteq \mathcal{A} \,\} \tag{3}$$

Lemma 28 in Appendix B shows that $(\mathcal{A})_{coh}$ is indeed a closure of $\mathcal{A}$ with respect to the operation of coherent intersection, hence the name. Coherent intersections give rise to the redundant frames. In fact, they are the only source of redundant frames.

Proposition 3. *A frame $P \in \Pi$ is redundant with respect to $\mathcal{A} \subseteq \Pi$ if and only if there is a subcollection $\mathcal{C} \subseteq \mathcal{A}$ such that P is a non-trivial coherent intersection of $\mathcal{C}$. Let axioms 1–7 hold for $c(\cdot)$. If $(U', \mathcal{A}')$ represents $c(\cdot)$, then the collection of frames $\mathcal{B}$ in the maximum representation $(U, \mathcal{B})$ of $c(\cdot)$ consists of all coherent intersections of subcollections $\mathcal{C} \subseteq \mathcal{A}'$.*

Corollary 1. *Let choice correspondence $c(\cdot)$ satisfying Axioms 1–7 be represented by $(U, \mathcal{A})$. If all frames $P \in \mathcal{A}$ are singletons, then the representation is unique up to a positive affine transformation of U.*

Proposition 3 characterizes all possible redundant frames. Moreover, it says that we can start with any representation $(U', \mathcal{A}')$ of a choice correspondence $c(\cdot)$ and find out the maximum collection of frames $\mathcal{B} = (\mathcal{A}')_{coh}$ by coherently intersecting frames in $\mathcal{A}'$. Notice that we can start with $\mathcal{A}' = \mathcal{A}$ being the minimum collection of frames, that is, $\mathcal{B} = (\mathcal{A})_{coh}$. In the special case when DM is a subjective expected utility maximizer with frame-dependent belief, $\mathcal{A}$ is a unique collection of frames (singleton beliefs) in the representation of $c(\cdot)$.

3.1 Identification of multi-multiple prior models in the literature

In Frick et al. (2019), the set of *relevant priors*—a convex hull of the priors that are used by DM to evaluate acts—is unique for the representation. This is true for our model as well: the set of relevant priors is a convex hull of the union of all frames in the minimum family[6]. Our model, therefore, suggests a stronger identification result, since our representation also exhibits a unique minimum family of sets of beliefs.

In Kopylov (2019), the analyst can identify the projections of sets of beliefs that account for the ambiguity relevant for given decision problems. Our model allows for the identification of each set of beliefs completely, but some redundant sets of beliefs may occur, although we fully characterize these redundancies.

[6]This follows from Proposition 3. Moreover, the set of priors that are used by DM to evaluate acts—and not just the convex hull of this set—is also unique in our model, since redundant sets of beliefs consist of intersections of sets of beliefs from the minimum family.

Both Frick et al. (2019) and Kopylov (2019) operate with a preference relation as a primitive[7]. Our model makes use of data on choices from larger menus as well, which is advantageous for the identification purposes—we clarify this point in Section 5.3.

Lu (2014) shows that the distribution of sets of beliefs is uniquely identified if the analyst knows frequencies of choices from doubleton menus consisting of an ex-ante lottery over Anscombe-Aumann acts and a constant act. His model, therefore, provides a superior identification result by extending the primitive.

Lu (2014)'s result comes at a cost of assuming linearity over the ex-ante mixtures of Anscombe-Aumnann acts, and implicitly assuming that the process of assignment of a menu is independent of the process governing the choice of the set of beliefs that DM uses. We, in contrast, can allow for a correlation between a menu and a set of beliefs as long as it is not perfect—we care only about the supports of the distributions.

Finally, Heller (2012) identifies a convex family of singleton beliefs uniquely from the choice correspondence, while our Corollary 1 shows that a non-convex family of singleton beliefs is also identified uniquely. Our model, even constrained to singleton beliefs, allows for a richer behavior. For instance, suppose the DM in our model has two frames: $P_1 = \{\mu_1\}$, that favors f to $0.5f + 0.5g$ to g, and $P_2 = \{\mu_2\}$, that favors g to $0.5f + 0.5g$ to f. Such DM chooses f and g, but not a compromise alternative $0.5f + 0.5g$ from menu $\{f, 0.5f + 0.5g, g\}$. In contrast, the DM in Heller (2012) should necessary choose a compromise alternative $0.5f + 0.5g$ if she chooses f and g, since this mixture is the best according to one of the intermediate beliefs $\lambda\mu_1 + (1-\lambda)\mu_2$ for some $\lambda \in (0,1)$.

4 Frame Sensitivity

In this section, we study the behavioral difference between agents who are *sensitive* to the framing of a choice problem—in the sense that they may change their decision depending on the frame that they have in mind—and agents who are not sensitive to the framing.

A Gilboa and Schmeidler (1989) maxmin model is a special case of a framed ambiguity model when the set of frames is singleton: $\mathcal{A} = \{P\}$. Intuitively, DM is able to hold the same perception of uncertainty independently of varying irrelevant details of the choice problem—either because framing does not matter for her, or because she is able to recognize the whole set of frames and connect them into a single frame. Formally:

[7] Kopylov (2019) considers also an extension of his model for choice functions.

DEFINITION 6. The DM is *not sensitive to framing* if her choice correspondence $c(\cdot)$ admits a framed ambiguity representation with a single frame.

If DM is not sensitive to framing, the analyst can use choices from binary menus (preference relation) to identify the corresponding set of beliefs uniquely, while in this paper we use data on choices from larger menus to identify a unique minimum family of frames. Can we achieve the same identification result using just a preference relation?

Formally, consider the following property:

Normality: $\forall A \in \mathcal{K} \quad c(A) = \{f \in A \mid f \in c(\{f, g\}) \; \forall g \in A\}$

Normality says that once the analyst knows choices from binary menus, the analyst can reconstruct the full choice correspondence by a straightforward way; consideration of larger menus in this case is not advantageous. We, therefore, are interested in cases when *Normality* holds for our DM.

Next, we want to shed some light on our key axioms—*Framed Uncertainty*, *No-C-Hedging* and *Indirect Ambiguity Aversion*—by considering their alternative versions that bear the same meaning for DM with one frame, but, possibly, different meaning for DM with multiple frames.

If DM has only one frame, her choice correspondence should satisfy *WARP*, which is strengthening of the *Framed Uncertainty* axiom.

The ambiguity aversion axiom for the maxmin model translates from the language of preference relation to the language of choice correspondence as follows: $f \in c(\{f, g\})$ implies $\lambda f + (1-\lambda)g \in c(\{\lambda f + (1-\lambda)g, g\})$. It is easy to see that this property holds for agent who operates with multiple frames as well. However, this property does not clarify what happens when DM chooses from non-doubleton menus. For instance, $f \in c(\{f, g\} \cup A)$ does not imply that $\lambda f + (1-\lambda)g \in c(\{\lambda f + (1-\lambda)g\} \cup \{A\})$, since DM might dislike g. We, therefore, require that g is also good enough: $g \in c(A \cup \{g\})$.

Direct Ambiguity Aversion: $\quad \forall A \in \mathcal{K} \;\; \forall f, g \in H \;\; \forall \lambda \in (0,1)$

$$f \in c(A \cup \{f\}) \text{ and } g \in c(A \cup \{g\}) \implies \lambda f + (1-\lambda)g \in c(A \cup \{\lambda f + (1-\lambda)g\})$$

A decision maker who satisfies *Direct Ambiguity Aversion* argues in the spirit of Axiom 5 (*Indirect Ambiguity Aversion*), but is able to connect her decisions under, possibly, different framing: if there is a reason to choose f versus acts in A and there is a reason to choose g versus acts in A, then there should be a reason to choose their mixture versus acts in A, since mixture hedges uncertainty in f and g. We are interested in cases when *Direct Ambiguity Aversion* holds for our agent.

Suppose agent has one frame, and she prefers act h to both act f and constant act p, i.e. $h \in c(\{h, f\})$ and $h \in c(\{p, h\})$. Then this agent respects a pairwise version of *No-C-Hedging*: it should be that $h \in c(\{\lambda f + (1-\lambda)p, h\})$. Can this property hold for DM with multiple frames as well? Formally,

Pairwise No C-Hedging: $\forall f, h \in H,\ \forall p \in H_0$

$$h \in c(\{h, f\}) \text{ and } h \in c(\{h, p\}) \implies h \in c(\{h, \lambda f + (1-\lambda)p\})$$

Intuitively, DM who satisfies *Pairwise No C-Hedging* argues in the spirit of Axiom 3 (*No C-Hedging*), but she can connect her decisions under different framing: if there is a reason to choose h versus f, and there is a reason to choose h versus p, then there should be a reason to choose h versus their mixture, since constant act p cannot effectively hedge against uncertainty in f.

The next proposition relates frame sensitivity to the violation of these properties:

Proposition 4. *Let choice correspondence $c(\cdot)$ satisfy Axioms 1–7. Then the following properties are equivalent for $c(\cdot)$: β, WARP, Normality, Direct Ambiguity Aversion, Pairwise No-C-Hedging, DM with choice correspondence $c(\cdot)$ being not sensitive to framing.*

Proposition 4 says that $c(\cdot)$ is inherently *not normal*: the only case when an analyst can reconstruct the whole choice correspondence using only information coming from the binary choices is when DM is not frame sensitive. Notice that an analyst can also verify frame sensitivity using only binary choices, since *Pairwise No-C-Hedging* operates with doubleton menus.

Intuitively, *Direct Ambiguity Aversion* and *Pairwise No-C-Hedging* are violated by a frame sensitive DM not because she doesn't respect the associated hedging reasoning, but because she fails to contemplate her choices under different frames, what is needed to build the premises of the corresponding arguments.

5 Connection of Frames and Consistency of Choices

Imagine two agents who agree on the ranking of prizes, say both prefer $x = \$100$ to $y = \$0$, but choose differently from menus consisting of acts that yield state-dependent lotteries over x and y. The ambiguity literature distinguishes two main reasons for that: different perception of the relevant uncertainty—that is, ambiguity—and different sentiment regarding the perceived uncertainty—that is, ambiguity attitude[8].

Our model allows to distinguish a third reason: difference in the sensitivity to framing. All else equal, two agents may have different ability to connect the same information about the environment into a decision procedure. In our comparative statics analysis, we focus on the situations when one of the agents is more capable of connecting various decision frames than another. Naturally, the choice of such decision maker exhibits more consistency and is less frame-dependent.

One way to connect the two points of view is to extend the aversion of ambiguity to frames as well, and evaluate everything from the worst point of view—when frames are singleton beliefs, this aggregation idea corresponds to *Default to Certainty* in Gilboa, Maccheroni, Marinacci, and Schmeidler (2010), and for non-singleton beliefs this is considered in Crès et al. (2011) and Hill (2011).

DEFINITION 7. Given a non-empty closed collection of frames $\mathcal{C}$, its *convex union* is a set of beliefs P given by

$$P \;=\; conv\left(\bigcup_{P'\in\mathcal{C}} P'\right) \tag{4}$$

Intuitively, an agent who contemplates a subset of frames $\mathcal{C}$ at a moment, can think that any prior in any set of beliefs is possible and be cautious toward this opportunity. An aggregate frame P is then a convex hull[9] of the union of all sets of beliefs in $\mathcal{C}$.

While cautious method of aggregation suggests to consider all possible priors, an optimistic decision maker can do the opposite: consider only priors that belong to every frame in the collection $\mathcal{C}$; that is, the intersection of frames. The choice behavior of such agent corresponds to contemplating a decision to choose an act versus a constant alternative within each frame as an argument in favor of this act—we call this type of reasoning,

[8] See, for instance, Ghirardato, Maccheroni, and Marinacci (2004).

[9] Non-convex sets of beliefs—as oppose to non-convex sets of frames—are not identifiable in our model; also, a decision maker who contemplates beliefs μ_1 and μ_2 within a given frame could contemplate as well an intermediate belief $\lambda\mu_1 + (1-\lambda)\mu_2$.

novel in the literature as far as we aware, an *optimistic learning* and analyze it in the end of the section. An obvious pitfall of this approach is that a collection $\mathcal{C}$ may have empty intersection. Some intersections of frames are coherent, and they play special role in our analysis, as one might guess from our identification results.

Finally, the agent can try to find a compromise between frames: interpret each frame as a point of view and assign a weight to it. For instance, if frames are singletons $P_1 = \{\mu_1\}$ and $P_2 = \{\mu_2\}$, then a singleton set $P = \lambda\{\mu_1\} + (1-\lambda)\{\mu_2\}$ corresponds to weighting[10] the first point of view—or expert's opinion—by λ, and second point of view by $1-\lambda$.

DEFINITION 8. Given a non-empty finite collection of frames $\mathcal{C} = \{P_1, \ldots, P_N\}$, its *convex combination* with respect to weights $\lambda \in \triangle(\{1,\ldots,N\})$ is a set of beliefs P given by

$$P \;=\; \sum_{i=1}^{N} \lambda_i P_i \;\equiv\; \left\{ \mu \in \triangle S \;\middle|\; \exists \mu_i \in P_i \text{ for } i = 1,\ldots,N : \; \mu = \sum_{i=1}^{N} \lambda_i \mu_i \right\} \tag{5}$$

A convex combination[11] P of a collection of frames $\mathcal{C}$ has a nice property that the value of an act calculated with respect P is a weighted average of its values with respect to each of the frames $\mathcal{C}$, and the weights are the same as used in the convex combination of frames. Thus, the idea of a compromise between different points of view works simultaneously for values of acts, and for beliefs as well.

In the remaining part of this section, we study the choices of—potentially frame sensitive—agent 1 who connects some of the frames of agent 2. We say that a collection of frames $\mathcal{C}$ is connected in a non-trivial way in a frame P by one of the discussed three ways, if $P \notin \mathcal{C}$.

5.1 Decisiveness

We start by considering a natural decisiveness relation.

DEFINITION 9. DM 1 with choice correspondence $c_1(\cdot)$ is *more decisive* than DM 2 with choice correspondence $c_2(\cdot)$ if $c_1 \subseteq c_2$.

[10] This way of aggregation different opinions is also considered in Crès et al. (2011) and Hill (2011).

[11] We consider finite collections for two reasons. First, a version of our results involving infinite collections lacks proofs at the moment. Second, infinite convex collection is a less intuitive object, since, for instance, we should require a selection of a prior in each frame to be measurable with respect to σ-algebra on $\mathcal{C}$.

Thus, we say that DM 1 is more decisive than DM 2 if she chooses among a subset of alternatives that could be chosen by DM 2. Proposition 3 provides us with necessary tools to characterize this comparative decisiveness relation:

Proposition 5. *Let $c_1(\cdot)$ and $c_2(\cdot)$ are represented by $(U_1, \mathcal{A}_1)$ and $(U_2, \mathcal{A}_2)$. Then DM 1 is more decisive than DM 2 if and only if U_1 is a positive affine transformation of U_2 and $\mathcal{A}_1 \subseteq (\mathcal{A}_2)_{coh}$.*

Recall that $(\mathcal{A}_2)_{coh}$ is a maximum collection of frames that represents $c_2(\cdot)$. Thus, Proposition 5 tells us that a more decisive agent is one who starts with the maximum collection of frames that the other agent may have and drops some of those. Therefore, the only non-trivial way how a more decisive agent may connect the frames of a less decision agent is by taking a coherent intersection of them.

5.2 Consistency

We proceed with a less demanding notion, which compares instances in which agents make the same choice consistently.

DEFINITION 10. DM 1 with choice correspondence $c_1(\cdot)$ is *more consistent* than DM 2 with choice correspondence $c_2(\cdot)$ if for all $A \in \mathcal{K}$ $\quad |c_2(A)| = 1 \implies |c_1(A)| = 1$.

It is easy to see that if DM 1 is more decisive than DM 2, then DM 1 is more consistent then DM 2, since $c_2(A) = \{f\} \implies c_1(A) = \{f\}$. Therefore, we know that if DM 1 considers coherent intersections of frames of DM 2, then she is more consistent. Note also that the binary relation on the set of framed ambiguity agents of given comparative consistency is transitive—as well as the binary relation given by the comparative decisiveness.

DEFINITION 11. Let $\mathcal{A} \subseteq \Pi$ be a non-empty collection of frames. Then its closure with respect to the operations of convex union, convex combination and coherent intersection is a minimum family of frames $\mathcal{A}^*$ that satisfies (i) $\mathcal{A} \subseteq \mathcal{A}^*$, and (ii) if P is either a convex union, or a convex combination, or a coherent intersection of a collection of frames $\mathcal{C} \subseteq \mathcal{A}^*$, then $P \in \mathcal{A}^*$.

Lemma 23 in Appendix A shows that the closure given by Definition 11 exists and is unique. As we have seen from Alice's example, connecting frames by taking convex unions, convex combinations or coherent intersections of frames (weakly) increases consistency of choices of DM. Our main result in this section is that, in fact, there operations of frame connection characterise the comparative consistency:

Theorem 3. *Let $c_1(\cdot)$ and $c_2(\cdot)$ be represented by $(U_1, \mathcal{A}_1)$ and $(U_2, \mathcal{A}_2)$ respectively, and $|\mathcal{A}_2| < \infty$. Then the following statements are equivalent:*

(i) DM 1 is more consistent than DM 2.

(ii) U_1 is a positive affine transformation of U_2, and $\mathcal{A}_1$ belongs to the closure of $\mathcal{A}_2$ with respect to operations of convex union, convex combination and coherent intersection.

(iii) U_1 is a positive affine transformation of U_2, and any $P \in \mathcal{A}_1$ is a coherent intersection of convex unions of convex combinations of frames in $\mathcal{A}_2$.

Note that Theorem 3 assumes additionally that DM 2 has a finite number of frames[12]. If $\mathcal{A}_2$ is infinite, one has to generalize the definition of the convex combination of frames to work with closed collections of frames, making it less intuitive[13]. We conjecture that after this generalization Theorem 3 will remain true[14].

5.3 Identification of frames from binary choices

Despite the fact that—by Proposition 4—the data on binary choices is not sufficient for reconstruction of the whole choice correspondence, our next result shows that this data is sufficient to tell if one DM is more consistent than another according to Definition 11.

Proposition 6. *Let $c_1(\cdot)$ and $c_2(\cdot)$ satisfy axioms 1–7. Then DM 1 is more consistent than DM 2 if and only if $\forall g, h \in H$ $c_1(\{f, g\}) \subseteq c_2(\{f, g\})$.*

While comparative consistency is defined via choices from arbitrary menus, there is a reason why this notion can be reduced to comparison of binary choices.

This binary formulation of comparative consistency allows us to use Theorem 3 to get identification result for a variant of our model when the primitive is preference relation—that is, data on binary choices—as oppose to the full choice correspondence. The next

[12]We acknowledge that we don't introduce the corresponding behavioral property.

[13]Weights λ generalize to the probability distribution on the Borel sigma-algebra of the given collection of frames, and selection of priors $\mu_i \in P_i$ generalizes to the measurable selection from the given collection.

[14]To prove that it would be enough to show that eq. (3) in Frick et al. (2019) works not only for functions $R^S \to R$, but also for functions $C(Y) \to R$ from the space of continuous functions on a compact set Y to R

result follows from the fact that $c_1(\cdot) = c_2(\cdot)$ for binary menus if and only if DM 1 is more consistent than DM 2 and the other way around.

Corollary 2. *Framed ambiguity models $(U, \mathcal{A})$ and $(U', \mathcal{A}')$ with $|\mathcal{A}|, |\mathcal{A}'| < \infty$ represent the same choice correspondence on the set of binary menus if and only if U and U' are positive affine transformations of each other, and closures of $\mathcal{A}$ and $\mathcal{A}'$ with respect to operations of convex union, convex combination and coherent intersection coincide.*

Corollary 2 tells us that observing choices only from binary menus leaves a big room for different representations. Moreover, in contrast to Theorem 2 for choice correspondence, an analyst may not be able to identify a unique minimum collection of frames that represents[15] a given preference relation[16], which is clear from the following example.

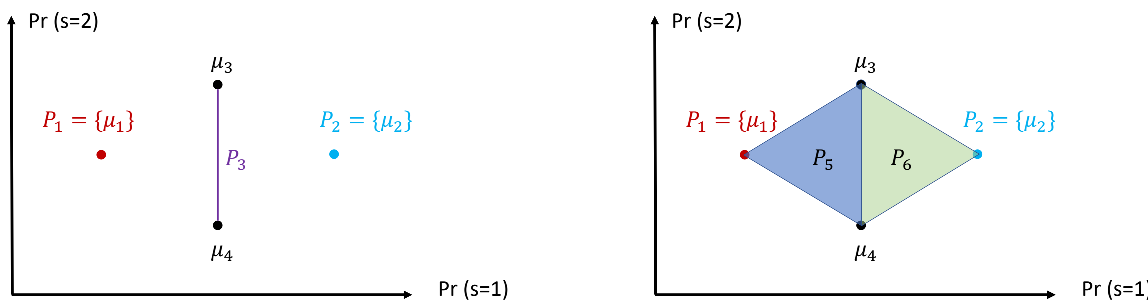

Figure 1

Let $|S| = 3$. Consider priors $\mu_1 = (0.1, 0.2, 0.7)$, $\mu_2 = (0.3, 0.2, 0.5)$, $\mu_3 = (0.2, 0.3, 0.5)$, $\mu_3 = (0.2, 0.1, 0.7)$ and sets of priors $P_1 = \{\mu_1\}$, $P_2 = \{\mu_2\}$, $P_3 = conv(\mu_3, \mu_4)$, $P_5 = conv(\mu_1, \mu_3, \mu_4)$, $P_6 = conv(\mu_2, \mu_3, \mu_4)$. Let the first family of frames be $\mathcal{A}_1 = \{P_1, P_2, P_3\}$, and the second family of frames be $\mathcal{A}_2 = \{P_1, P_2, P_5, P_6\}$. Note that $P_5 = conv(P_1 \cup P_3)$, and $P_6 = conv(P_2 \cup P_3)$, and $P_3 = P_5 \cap P_6$, where the intersection of P_5 and P_6 is coherent. Hence, the closures of $\mathcal{A}_1$ and $\mathcal{A}_2$ with respect to the operations of convex union, convex combination and coherent intersection coincide.

It is straightforward to see that the corresponding choice over binary menus does not have a minimum representation, since in the model with the set of frames $\mathcal{A}_3 = \mathcal{A}_1 \cap \mathcal{A}_2 = \{P_1, P_2\}$, the DM's beliefs regarding $s = 2$ is fixed: $\mu(2) = \mu_1(2) = \mu_2(2) = 0.2$, while the original model allows for $\mu_3(2) = 0.3$ and $\mu_4(2) = 0.1$.

[15] Nothing changes for the identification of the vNM expected utility function, since the restriction $c_0(\cdot)$ of choice correspondence on the set $\mathcal{K}_0$ of menus consisting of constant acts satisfies *WARP*.

[16] Where we define preference relation via choice from binary menus by $f \succeq g$ if $f \in c(\{f, g\})$.

In the example above, if an analyst observes choices from non-binary menus as well, then $(U, \mathcal{A}_1)$ and $(U, \mathcal{A}_2)$ represent different choice correspondences $c_1(\cdot)$ and $c_2(\cdot)$. By Proposition 3, $\mathcal{A}_1$ is the unique family of frames representing $c_1(\cdot)$. Choice correspondence $c_2(\cdot)$ admits representations with two different families of frames: $\mathcal{A}_2$ is the minimum family, and $(\mathcal{A}_2)_{coh} = \mathcal{A}_2 \cup P_3$ is the maximum family.

5.4 Application to the Social Choice Theory

Our model has a population interpretation: we can think about a group of people that agree on the ranking of prizes (share the same utility), are averse to the ambiguity, but disagree on beliefs. Suppose they want to aggregate their judgements such that the resulting rule still respects ambiguity aversion, and a natural *Unanimity* criterion:

DEFINITION 12. Let $\{\succeq_i\}_{i=1,...,N}$ and $\succeq$ be preference relations on H. We say that $\succeq$ satisfies *Unanimity* with respect to $\{\succeq_i\}_{i=1,...,N}$ if for all $f, g \in H$ $[f \succeq_i g \; \forall i = 1, ..., N] \implies f \succeq g$.

Then Theorem 3 helps us to characterize the set of such rules:

Corollary 3. *Let $\{\succeq_i\}_{i=1,...,N}$ and $\succeq$ be preference relations on H that satisfy axioms of Gilboa and Schmeidler (1989) and have maxmin representations with the same vNM expected utility function, but different sets of beliefs $\{P_i\}_{i=1,...,N}$ and P. Then $\succeq$ satisfies Unanimity with respect to $\{\succeq_i\}_{i=1,...,N}$ if and only if P is a coherent closure of convex unions of convex combinations of $\{P_1, ..., P_N\}$.*

Notice also that taking arbitrary number and order of operations of coherent intersection, convex union and convex combination of $\{P_1, ..., P_N\}$ always produces a rule that satisfies Unanimity.

5.4.1 Aggregation of ambiguity averse opinions in the literature

In Crès et al. (2011), the authors study the aggregation of the opinions of a group of ambiguity-averse experts. They impose a stronger *Expert Uncertainty Aversion* axiom which is a bit less straightforward then the *Unanimity* axiom. In terms of our result, their aggregation procedure is equivalent to taking convex unions and convex combinations of sets of beliefs, but not coherent intersections. We show that usage of all three of these operations characterises the aggregation procedure that satisfies the unanimity criterion. To get this result, we use some of the instruments from proofs in Crès et al. (2011) and Frick et al. (2019).

In Hill (2011), the author characterizes the same aggregation rule as Crès et al. (2011) in terms of the unanimity criterion plus the requirement that an aggregation rule should remain the same across all profiles of preferences. In terms of our model, his additional requirement rules out coherent intersections primarily because not all intersections of sets of beliefs are coherent, hence if a rule includes a coherent intersection, it cannot be applied for all profiles of preferences.

5.5 Optimistic Learning

Let us return to the idea that DM may consider frames as pieces of information about priors that cannot be correct. Can we characterise this type of learning via observed choices?

Suppose the agent considers act $h = 0.5f + 0.5g$; frame P_1 is favorable for act f, while frame P_2 is favorable for act g. DM may then optimistically evaluate act h as one half of the value of f in frame P_1 plus one half of the value of g on frame P_2. Moreover, if the agent is sophisticated enough, she may also consider consider, for instance, a decomposition[17] $\lambda h + (1-\lambda)q = \sigma f' + (1-\sigma)g'$ for some constant act q, convex wights λ, σ and acts f', g', and figure out the value of h by considering the most favorable frames for acts f', g' and then using the *C-Independence* principle. The next definition formalizes this idea.

DEFINITION 13. DM 1 optimistically learns from DM 2 if there exist $q \in H_0$ such that for all $f \in H$ there exist $\lambda \in (0,1]$, $f_1, ..., f_k \in H$ and $\sigma \in \triangle(\{1, ..., k\})$ such that:

(i) $\lambda f + (1-\lambda)q \;=\; \sum_{i=1}^{k} \sigma_i f_i$

(ii) $\lambda f + (1-\lambda)q \in c_1(\{\lambda f + (1-\lambda)q, p\}) \iff f_i \in c_2(\{f_i, p\}) \; \forall i = 1, ..., N \quad \forall p \in H_0$

Intuitively, DM 2 interprets $f_i \in c_2(\{f_i, p\})$ as an argument in favor of f_i versus constant act p, and then optimistically claims that these arguments should work for the mixture of acts f_i as well. To present our last result, we need one of the two following technical assumptions.

[17]The reason why we need this extra step is that the set of acts H is compact. Thus, for instance, if act f is an extreme point of H in the space of utilities, it does not have a non-trivial decomposition.

CONDITION 1. A family of frames $\mathcal{A}$ is finite, its intersection is non-empty, and each $P \in \mathcal{A}$ is polyhedral.

Condition 1 says that each $P \in \mathcal{A}$ can be expressed via a finite number of linear inequalities[18]. An equivalent formulation is that each $P \in \mathcal{A}$ is a polytope—that is, there are finitely many priors $\mu \in P$ that DM uses to evaluate acts with respect to P. If we need to consider more general sets of priors, we can use the following condition, where we denote by $ri(P)$ the relative interior of P:

CONDITION 2. A family of frames $\mathcal{A}$ is finite, and $\bigcap_{P \in \mathcal{A}} ri(P)$ is non-empty.

Intuitively, Condition 2 says that the intersection of frames does not cause the ambiguity existing in each frame in some dimension to vanish. The following proposition make use of results from Convex Analysis[19]:

Proposition 7. *Let choice correspondences $c_1(\cdot)$ and $c_2(\cdot)$ be represented $(U_1, \{P_1\})$ and $(U_2, \mathcal{A}_2)$, and either Condition 1 or Condition 2 holds for $\mathcal{A}_2$. Then DM 1 optimistically learns from DM 2 if and only if U_1 is a positive affine transformation of U_2, and $P_1 = \bigcap_{P \in \mathcal{A}_2} P$.*

DM 1 in Proposition 7 has only one frame in mind, thus $c_1(\cdot)$ satisfies WARP and is represented by the maxmin model with vNM expected utility $U_1(\cdot)$ and set of beliefs P_1. Proposition 7 then says that if DM 1 becomes rational by optimistically connecting arguments $f_i \in c_2(\{f_i, p\})$ ("f_i is weakly better than p according to some frame"), then it is equivalent to learning about priors that cannot be correct—that is, intersecting sets of beliefs.

[18]And equations as well, since $a = b$ is equivalent to $a \geq b$ and $a \leq b$.

[19]Corollary 16.4.1 and Theorem 20.1 in Rockafellar (1970).

Appendix A

In this section, we provide proofs of all main statements—theorems, propositions and corollaries—except of Proposition 1, which is proven, among some other statements, in Appendix B.

Proof of Theorem 1

The proof that axioms follow from the representation is straightforward. We omit it except of the proof for the *Continuity* axiom.

Lemma 1. *Function $W: \Pi \times H \to \mathbb{R}$ given by eq. (2) is continuous.*

Proof. Follows from the Berge theorem (Theorem of Maximum). □

Assume choice correspondence $c(\cdot)$ has representation $(U, \mathcal{A})$. Consider $(A^k, f^k) \longrightarrow (A, f)$ such that $f^k \in c(A^k)$ for all $k = 1, 2, \ldots$. Then $f^k \in A^k$, and $\rho(\{f\}, A) \le d(f, f^k) + \rho(\{f^k\}, A) \le d(f, f^k) + \rho(A^k, A) \longrightarrow 0$, hence $f \in cl(A) = A$. Since $f^k \in c(A^k)$ for $k = 1, 2, \ldots$, then there exists $P^k \in \mathcal{A}$ such that $W_{P^k}(f^k) \ge W_P(g)$ for all $g \in A^k$. Since Π is compact, there is a convergent subsequence $P^{k_m} \longrightarrow P \in \Pi$. Since $\mathcal{A}$ is closed, $P \in \mathcal{A}$. Since $A^{k_m} \longrightarrow A$, for arbitrary $g \in A$ there exists $g^{k_m} \in A^{k_m}$ such that $g^{k_m} \longrightarrow g$. Then $W_{P^{k_m}}(f^{k_m}) \ge W_{P^{k_m}}(g^{k_m})$ for all $m = 1, 2, \ldots$. Since function $(P, f, g) \to W_P(f) - W_P(g)$ is continuous by Lemma 1, we conclude that $W_P(f) \ge W_P(g)$. Since this holds for all $g \in A$, then $f \in c(A)$. Thus, *Continuity* holds for $c(\cdot)$.

Assume now that choice correspondence $c(\cdot)$ satisfies Axioms 1-7. We show that $c(\cdot)$ admits a framed ambiguity representation.

Lemma 2. *Continuity (Axiom 6) implies axioms defined below:*
Act Continuity: $\forall A \in \mathcal{K}$ $\{f \in H \mid f \in c(A \cup \{f\})\}$ is closed
Menu Continuity: $\forall f \in H$ $\{A \in \mathcal{K} \mid f \in c(A)\}$ is closed

Proof. *Act Continuity* follows from $f^n \to f \implies (A \cup \{f^n\}, f^n) \longrightarrow (A \cup \{f\}, f)$. *Menu Continuity* follows from $A^n \longrightarrow A \implies (A^n, f) \to (A, f)$. □

We say that the restriction of choice correspondence c_0 on the set of compact menus of constant acts $\mathcal{K}_0$ has an expected utility representation, if there is a vNM expected utility

$U: \triangle X \to R$ such that $\forall A \in \mathcal{K}_0 \ \ [(p,...,p) \in c(A) \iff [U(p) \geq U(q) \ \ \forall q \in A]]$. Note that if $c(\cdot)$ satisfies Axioms 1-7, so does $c_0(\cdot)$.

Lemma 3. *Given Axioms 1-7, c_0 has a non-degenerate expected utility representation that is unique up to a positive affine transformation.*

Proof. Consider binary relation $\succeq_{c_0} \subseteq H_0 \times H_0$ given by $p \succeq_{c_0} q$ if $p \in c_0(\{p,q\})$. Axioms α and *C-β* imply *WARP* for c_0, thus $\succeq_{c_0}$ is complete and transitive. The *C-Independence* for $c_0(\cdot)$ implies Independence for $\succeq_{c_0}$. *Act Continuity* implies $\{q \in H \mid p \succeq_{c_0} q\}$ is closed, and *Menu Continuity* implies $\{q \in H \mid p \preceq_{c_0} q\}$ is closed $\forall p \in H_0$. *C-Non-Degeneracy* implies $p \not\succeq q$ for some p,q. The Lemma follows from the Mixture Space Theorem. □

Lemma 3 allows us to identify $U(\cdot)$ from $c(\cdot)$. Since X and $\triangle X$ are compact, there is the best prize $x \in X$ and the worst prize $y \in Y$. Abusing notations, we denote $\delta_x \in \triangle X$ by x and δ_y by y. We normalize $U(\cdot)$ by $1 = U(x) \geq U(p) \geq U(y) = 0$ for all $p \in \triangle X$.

Lemma 4. *If $c(\cdot)$ satisfies Axioms 1-7, then it satisfies the following axiom:*
Monotonicity: $\forall f,g \in H \quad f(s) \in c(\{f(s),g(s)\}) \ \forall s \in S$ implies
$\forall A \in \mathcal{K} \quad$ *(i)* $[g \in c(A) \implies f \in c(A \cup \{f\})]$ *and (ii)* $[f \in A \implies c(A) \subseteq c(A \cup \{g\})]$

Proof. For all menus $A \in \mathcal{K}$ and acts $f \in H$ denote

$$A_{0.5} \equiv 0.5A + 0.5(0.5x + 0.5y) \quad , \quad f_{0.5} \equiv 0.5f + 0.5(0.5x + 0.5y)$$

Assume $f(s) \in c(\{f(s), g(s)\}) \ \ \forall s \in S$. Then by Lemma 3, $U(f(s)) \geq U(g(s)) \ \ \forall s \in S$.
Proof of (i) Let $g \in c(A)$, then by *C-Independence* $g_{0.5} \in c(A_{0.5})$. Consider $f^n_{0.5} \equiv (1-1/n) f_{0.5} + (1/n)x$ for $n = 1,2,....$ Then $\forall s \in S$ we have:

$$U(f^n_{0.5}(s)) = (1-1/n)(0.5U(f(s)) + 0.25) + 1/n >$$
$$0.5U(f(s)) + 0.25 = U(f_{0.5}(s)) \geq U(g_{0.5}(s))$$

By Lemma 3, $g_{0.5}(s) \notin c(\{g_{0.5}(s), f^n_{0.5}(s)\}) \ \forall s \in S$. *Strict Monotonicity* implies $g_{0.5} \notin c(\{g_{0.5}, f^n_{0.5}\})$, then by α we have $g_{0.5} \notin c(A_{0.5} \cup \{f^n_{0.5}\})$. By *Aizerman's Property* $f^n_{0.5} \in c(\{A_{0.5} \cup f^n_{0.5}\})$. Since $f^n_{0.5} \longrightarrow f_{0.5}$, by *Act Continuity* $f_{0.5} \in c(A_{0.5} \cup \{f_{0.5}\})$. By *C-Independence* $f \in c(A \cup \{f\})$.
Proof of (ii) Assume $f \in A$ and consider $h \in c(A)$. By *C-Independence* $h_{0.5} \in c(A_{0.5})$. Consider $g^n_{0.5} \equiv (1-1/n) f_{0.5} + (1/n)y$ for $n = 1,2,....$ Then $\forall s \in S$ we have:

$$U(g^n_{0.5}(s)) = (1-1/n)(0.5U(f(s)) + 0.25) <$$
$$0.5U(g(s)) + 0.25 = U(g_{0.5}(s)) \leq U(f_{0.5}(s))$$

By Lemma 3, $g^n_{0.5}(s) \notin c(\{g^n_{0.5}(s), f_{0.5}(s)\})$ for all $s \in S$, by *Strict Monotonicity* $g^n_{0.5} \notin c(\{g^n_{0.5}, f_{0.5}\})$, by α we have $g^n_{0.5} \notin c(A_{0.5} \cup \{g^n_{0.5}\})$. By *Aizerman's Property* $h_{0.5} \in c(A_{0.5} \cup \{g^n_{0.5}\})$. Next, $A_{0.5} \cup \{g^n_{0.5}\} \longrightarrow A_{0.5} \cup \{g_{0.5}\}$, then by *Menu Continuity* $h_{0.5} \in c(A_{0.5} \cup \{g_{0.5}\})$. Finally, by *C-Independence* $h \in c(A \cup \{g\})$. □

Define binary relations $\succeq^*, \sim \ \subseteq H \times H$ by

$$f \succeq^* g \ \text{ if } \ f(s) \in c(\{f(s), g(s)\}) \ \forall s \in S \quad , \quad f \sim g \ \text{ if } \ [f \succeq^* g \text{ and } g \succeq^* f]$$

Using Lemma 3, we get $f \succeq^* g \iff U(f(s)) \geq U(g(s)) \forall s \in S$ and $f \sim g \iff U(f(s)) = U(g(s)) \ \forall s \in S$. Thus, $\sim$ is an equivalence relation on the set of acts. Consider a subset of acts that yield a convex combination of x and y at each state of the world:

$$\widehat{H} \ \equiv \ \left\{ h \in H \mid \exists \mathbf{z} \in [0,1]^{|S|} \ : \ f(s) = z_s x + (1 - z_s) y \ \forall s \in S \right\}$$

For each act $f \in H$ construct a canonical act $\widehat{f} \in \widehat{H}$ as follows:

$$\widehat{f}(s) \ \equiv \ U(f(s))x + (1 - U(f(s)))y$$

Act $\widehat{f}$ is well-defined, since $0 \leq U(f(s)) \leq 1$. We can see that if $f \sim g$, then $\widehat{f} = \widehat{g}$ and also $\widehat{f} = \widehat{\widehat{f}}$, since $U(x) = 1$, $U(y) = 0$. Thus, a canonical act $\widehat{f}$ represents an equivalence class of relation $\sim$. Given menu $A \in \mathcal{K}$, define canonical menu $\widehat{A}$ to be the set consisting of canonical acts corresponding to all equivalence classes of relation $\sim$ in A:

$$\widehat{A} \ \equiv \ \{ \ h \in \widehat{H} \mid \exists f \in A \ : \ f \sim h \ \} \tag{6}$$

Lemma 5. *Assume Axioms 1-7 hold. Then for any menu $A \in \mathcal{K}$ the corresponding canonical menu is a menu, i.e. $\widehat{A} \in \mathcal{K}$. Moreover,*

$$f \in c(A) \iff \widehat{f} \in c(\widehat{A})$$

Proof. We first show that $\widehat{A}$ is a menu. Consider an arbitrary sequence $\{\widehat{f^n}\} \subseteq \widehat{A}$, where $\{f^n\} \subseteq A$. Since A is compact, there is a convergent subsequence $f^{n_k} \longrightarrow f \in A$. Therefore, by continuity of $U(\cdot)$, $U(f^{n_k}(s)) \longrightarrow U(f(s))$ for all s, hence $\widehat{f^{n_k}} \longrightarrow \widehat{f} \in \widehat{A}$. Since an arbitrary sequence in $\widehat{A}$ contains a convergent subsequence, $\widehat{A}$ is compact, and since $\widehat{A}$ non-empty, $\widehat{A} \in \mathcal{K}$.

Claim 1. *Let $\{h_1, ..., h_k\} \subset H$. Assume $\forall i \ \exists g_i \in A$: $g_i \sim h_i$. Then:*

$$c(A \cup \{h_1, ..., h_k\}) \ = \ c(A) \cup \{h \in \{h_1, ..., h_k\} \mid \exists g \in c(A) : h \sim g\}$$

Proof of Claim 1. Let $h \sim g \in B$. Case (a): $g \notin c(B)$. If $h \in c(B \cup \{h\})$, then by *Monotonicity*, $g \in c(B \cup \{h\})$, and by α, $g \in c(B)$, contradiction. Therefore, $h \notin c(B \cup \{h\})$, and by *Aizerman's Property* and α, $c(B \cup \{h\}) = c(B)$.

Case (b): $g \in c(B)$. If $h \notin c(B \cup \{h\})$, then by *Aizerman's Property*, $g \in c(B \cup \{h\})$, and by *Monotonicity*, $h \in c(B \cup \{h\})$. Next, since $g \succeq^* h_{i+1}$, by *Monotonicity*, $c(B) \subseteq c(B \cup \{h\})$. By α and the above calculations, $c(B \cup \{h\}) = c(B) \cup \{h\}$.

Note that by *Monotonicity* and α, if $h_{k+1} \sim h_j$ for $j \leq k$ and $h_j \in c(A \cup \{h_1, ..., h_k\})$, then $g_j \in c(A \cup \{h_1, ..., h_k\})$ for $g_j \in A$. A straightforward induction argument that uses cases (a) and (b) completes the proof of the Claim. □

For any menu A, since it is compact, there exist a finite set of acts F_n such that $F_n \subseteq A \subseteq \bigcup_{f \in F_n} \{g \in H | \, d(g, f) < 1/n\}$. Denote by $F^n \equiv \bigcup_{m=1}^{n} F_m$, $\widehat{F^n} = \{h \in \widehat{H} \mid \exists f \in F^n : h = \widehat{f}\}$.

Claim 2. *For any menu A: (i)* $F^n \cup \widehat{A} \longrightarrow A \cup \widehat{A}$, *(ii)* $A \cup \widehat{F^n} \longrightarrow A \cup \hat{A}$.

Proof of Claim 2. Statement (i) is obvious, we prove statement (ii). We know that $\widehat{F^n} \subseteq \hat{A}$. Consider $h \in \widehat{A}$. Then $\exists f \in A$ such that $f \sim h$. Since each of F_n induces a cover of A by balls of radius $1/n$, for any n there is $f^n \in F_n$, $d(f, f^n) < 1/n$, hence $f^n \longrightarrow f$; note also that $\{\widehat{f^m}\}_{m=1}^n \subseteq \widehat{F^n}$. By continuity of $U(\cdot)$, $U(f^n(s)) \longrightarrow U(f(s))$, hence $\widehat{f^n} \longrightarrow \widehat{f} = h$. Since $U(\cdot)$ is uniformly continuous on a compact set $\triangle X$, and $A \cup \hat{F}^n \subseteq A \cup \widehat{A}$, then

$$\rho(A \cup \widehat{F^n}, A \cup \widehat{A}) \; = \; \max_{h \in A \cup \widehat{A}} \; \min_{g \in A \cup \widehat{F^n}} d(g, h) \; \leq \; \max_{h \in \widehat{A}} \; \min_{g \in \widehat{F^n}} d(g, h) \; \longrightarrow \; 0$$ □

Assume $f \in c(A)$. Without loss of generality (WLOG), $f \in F_1 \subseteq F^n$. By Claim 1, $f \in c(A \cup F^n)$ for all n. By *Menu Continuity* and Claim 2, $f \in c(A \cup \widehat{A})$. By *Monotonicity*, $\widehat{f} \in c(A \cup \widehat{A})$. By α, $\widehat{f} \in c(\widehat{A})$.

Assume $\widehat{f} \in c(\widehat{A})$. WLOG, $f \in F_1 \subseteq F^n$. By Claim 1, $\widehat{f} \in c(\widehat{A} \cup F^n)$. By *Menu Continuity* and Claim 2, $\widehat{f} \in c(\widehat{A} \cup A)$. By *Monotonicity*, $f \in c(\widehat{A} \cup A)$. By α, $f \in c(A)$. □

Lemma 5 allows us to focus on finding a representation for the choice correspondence restricted to the subset of canonical menus. Once we find a framed ambiguity representation of the restriction, it will automatically represent the whole choice correspondence as well, since $U(f(s)) = U(\widehat{f}(s)))$ for all $s \in S$ and $f \in H$. Thus, we assume for now that $H = \widehat{H}$. The set of canonical acts is isomorphic to the $|S|$- dimensional hypercube $[0,1]^{|S|}$ via identification of $\mathbf{z} \in [0,1]^{|S|}$ with $f \in \widehat{H}$ by $z_s = f(s)(x)$. We will use both $\mathbf{z} \in [0,1]^{|S|}$

and $f \in H$ as generic acts in the subsequent calculations. Denote $\iota = (1,...,1) \in R^{|S|}$, then act $\mathbf{z} \in [0,1]^{|S|}$ is constant if and only if $\mathbf{z} = \bar{z}\iota$ for some $\bar{z} \in [0,1]$. Next, the premise of the *Strict Monotonicity* Axiom for acts $\mathbf{z},\mathbf{z}'$ looks as $\mathbf{z} >> \mathbf{z}'$, and the premise of the *Monotonicity* Axiom looks as $\mathbf{z} \geq \mathbf{z}'$.

Lemma 6. *Let Menu Continuity hold. Then* $\forall f \in H, \forall A \in \mathcal{K}$ *if* $f \in c(A)$, *then there is a menu* $D \in \mathcal{K}$ *such that* $A \subseteq D$, $f \in c(D)$ *and* D *is maximal subject to this; that is, for any menu* $D' \neq D$ *such that* $D \subseteq D'$, $f \notin c(D')$.

Proof. Consider the set of menus that contain A such that f is chosen:

$$\mathcal{Q}(A,f) \equiv \{B \in \mathcal{K} \mid f \in c(B) \text{ and } A \subseteq B\},$$

partially ordered by set inclusion. Notice that $A \in \mathcal{Q}(A,f) \neq \emptyset$. Let $\mathcal{T}$ be an arbitrary non-empty chain (totally ordered subset) in $\mathcal{Q}(f,A)$. We argue that $\mathcal{T}$ has an upper bound in $\mathcal{Q}(f,A)$ equal to

$$C = cl\left(\bigcup_{B \in \mathcal{T}} B\right),$$

where closure is taken with respect to the topology on the space of acts H, and that $C \in \mathcal{Q}(A,f)$. Notice that $H \simeq [0,1]^{|S|}$ is compact. C is a closed subset of a compact set, hence it is compact, and $C \in \mathcal{K}$. Next, $B \subseteq C$ for all $B \in \mathcal{T}$ by construction. It remains to show that $f \in c(C)$. Let us show that for any $\epsilon > 0$ there is $B \in \mathcal{T}$ such that for any $B' \in \mathcal{T}$ such that $B \subseteq B'$ it is true that $\rho(B',C) < \epsilon$. Consider an open cover of C by a family of open balls of radius $\epsilon/3$ centered at all points of C:

$$\mathcal{G} = \{\{g \in H \mid \rho(g,f) < \epsilon/3\}_f\}_{f \in C}$$

Since C is a compact set, there is a finite subcover

$$\{\{g \in H \mid \rho(g,f_i) < \epsilon/3\}\}_{i=1}^N \subseteq \mathcal{G} \quad , \quad C \subseteq \bigcup_{i \in \{1,...,N\}} \{g \in H \mid \rho(g,f_i) < \epsilon/3\}$$

Recall that $B \subseteq C$ for any $B \in \mathcal{T}$, hence

$$\rho(B,C) = \sup_{g \in C}\left(\inf_{h \in B} d(g,h)\right) \leq \epsilon/3 + \max_{i \in \{1,...,N\}}\left(\inf_{h \in B} d(f_i,h)\right)$$

Since $f_i \in C$, for any $i = 1,...,N$ there is $B_i \in \mathcal{T}$ such that there is $g_i \in B_i$ such that $d(f_i,g_i) < \epsilon/3$. Since $\mathcal{T}$ is totally ordered by set inclusion, there is $\widetilde{B} \in \{B_1,...,B_N\}$ such that $B_i \subseteq \widetilde{B}$ for all $i = 1,...,N$ (every totally ordered finite set has a maximum element). Therefore, $\{g_1,...,g_n\} \subseteq \widetilde{B}$, hence $\rho(\widetilde{B}, C) < 2\epsilon/3$. Since we can find such $\widetilde{B} \in \mathcal{T}$ for arbitrary $\epsilon > 0$, it follows that $C \in cl(\mathcal{B})$. Therefore, by *Menu Continuity* $f \in c(C)$. We proved that every non-empty chain in $\mathcal{Q}(A, f)$ has an upper bound in $\mathcal{Q}(A, f)$, hence by Zorn's Lemma $\mathcal{Q}(A, f)$ has at least one maximal element D. ■

For a given act $\mathbf{z} \in H$ denote

$$\mathcal{L}(\mathbf{z}) \;\equiv\; \{\, A \in \mathcal{K} \mid \mathbf{z} \in c(A) \text{ and } [A \subseteq A',\ \mathbf{z} \in c(A') \implies A' = A]\,\} \tag{7}$$

In other words, $\mathcal{L}(\mathbf{z})$ is a set of menus that are maximal with respect to $\mathbf{z}$ being chosen from a menu. Lemma 6 implies that $\mathcal{L}(\mathbf{z}) \neq \varnothing$ for any $\mathbf{z} \in H$, since $\mathbf{z} \in c(\{\mathbf{z}\})$.
For $\lambda \in (0, 1]$ denote

$$Z_\lambda \;=\; \lambda[0,1]^{|S|} \;+\; (1-\lambda)\{0.5\iota\} \tag{8}$$

Thus, Z_λ is a hypercube homothetic to $Z_1 = H = [0,1]^{|S|}$ with the same center point 0.5ι. Similarly, for arbitrary menu $B \in \mathcal{K}$ denote $B_\lambda \;=\; \lambda B \;+\; (1-\lambda)\{0.5\iota\}$. Notice that $B_\lambda \subseteq Z_\lambda$. Next, by *C-Independence* $c(B_\lambda) \;=\; \lambda c(B) + (1-\lambda)\{0.5\iota\}$. Therefore, we are able to reconstruct the full choice correspondence knowing choice from menus $B_\lambda \subseteq Z_\lambda$. For arbitrary $\mathbf{z}$ and maximal menu $A \in \mathcal{L}(\mathbf{z})$ define menu's "certainty equivalent"

$$a(A) \;=\; \max\,\{b \in [0,1] \mid b \cdot \iota \in A\} \tag{9}$$

Function $a(.)$ is well-defined, since by *Monotonicity* any maximal menu contains the worst act $(0,...,0)$, and compactness of A insures the existence of maximum. Consider the set of menus that are maximal for some act $\mathbf{z} \in Z_{0.2}$:

$$\mathcal{D}_{0.2} = \{A \in \mathcal{K} \mid \exists \mathbf{z} \in Z_{0.2}:\ A \in \mathcal{L}(\mathbf{z})\} \tag{10}$$

Since $0.4\iota \le \mathbf{z} \le 0.6\iota$ for all $\mathbf{z} \in Z_{0.2}$, by *Monotonicity* $a(A) \ge 0.4$, and by *Strict Monotonicity* $a(A) \le 0.6$ for any $A \in \mathcal{D}_{0.2}$. For a vector $\mathbf{v} \in R^{|S|}$ denote $|\mathbf{v}| = \max_{s \in S} |v_s|$. We do not use *Continuity* in calculations involving $|\cdot|$, thus we don't care about the compatibility of $|\cdot|$ with metric d on H. Let

$$O \;=\; \left\{ \mathbf{v} \in R^{|S|} \;\middle|\; \sum_{s \in S} v_s = 0 \text{ and } |\mathbf{v}| = 1 \right\} \tag{11}$$

Define function $J:\ \mathcal{D}_{0.2} \times O \to R$ by the following formula:

$$J_A(\mathbf{v}) \;\equiv\; -\max\,\{b \in [-1,1] \mid a(A)\iota + 0.1 \cdot \mathbf{v} + 0.1 \cdot b \cdot \iota \in A\} \tag{12}$$

Notice that $b \in [-1,1]$ and $\mathbf{v} \in O$, thus for $\mathbf{z} = a(A)\iota + 0.1 \cdot \mathbf{v} + 0.1 \cdot b \cdot \iota$ we have $0.2\iota \leq \mathbf{z} \leq 0.8\iota$, thus $\mathbf{z} \in Z_{0.6} \subset Z_1 = H$, and $\mathbf{z}$ is an act; and the maximum exists, since A is compact. Next note that $a(A)\iota + 0.1 \cdot \mathbf{v} - 0.1\iota \leq a(A)\iota$, hence by *Monotonicity* $a(A)\iota + 0.1 \cdot \mathbf{v} - 0.1\iota \in A$. Similarly, for any $\epsilon > 0$ we have $a(A)\iota + 0.1 \cdot \mathbf{v} + 0.1 \cdot (1+\epsilon)\iota >> a(A)\iota$, hence by *Strict Monotonicity* $a(A)\iota + 0.1 \cdot \mathbf{v} + 0.1 \cdot (1+\epsilon)\iota \notin A$. Thus, there is no loss of generality to consider $b \in [-1,1]$ versus $b \in R$ in the definition of $J_{\cdot}(\cdot)$. Therefore, for any $\mathbf{v} \in O$, $A \in \mathcal{D}_{0.2}$ and any b such that $a(A)\iota + 0.1 \cdot \mathbf{v} + 0.1 \cdot b \cdot \iota \in Z_1$ is an act, using *Monotonicity* and maximality of $A \in \mathcal{D}_{0.2}$, we may conclude:

$$b \leq -J_A(\mathbf{v}) \iff a(A)\iota + 0.1 \cdot \mathbf{v} + 0.1 \cdot b \cdot \iota \in A \tag{13}$$

For $\mathbf{z} \in R^{|S|}$ consider its unique decomposition

$$\mathbf{z} = \bar{z} \cdot \iota + \mathbf{z}_\perp \quad , \quad \text{where} \quad \bar{z} = |S|^{-1} \cdot \sum_{s \in S} (\mathbf{z})_s \quad , \quad \text{and} \quad \mathbf{z}_\perp = \mathbf{z} - \bar{z} \cdot \iota \tag{14}$$

Then $\sum_{s \in S} (\mathbf{z}_\perp)_s = 0$. Define function $I: \mathcal{D}_{0.2} \times R^{|S|} \to R$ as follows:

$$I_A(z) \equiv \begin{cases} \bar{z} + |\mathbf{z}_\perp| \cdot J_A\left(\frac{\mathbf{z}_\perp}{|\mathbf{z}_\perp|}\right) & \text{if} \quad \mathbf{z}_\perp \neq \mathbf{0} \\ \bar{z} & \text{if} \quad \mathbf{z}_\perp = \mathbf{0} \end{cases} \tag{15}$$

The following lemma is an analogue of Lemma 3.3. in Gilboa and Schmeidler (1989).

Lemma 7. *Let Axioms 1-7 hold. Then for all $A \in \mathcal{D}_{0.2}$ and all $\mathbf{z} \in Z_{0.4}$ we have $\mathbf{z} \in A \iff I_A(\mathbf{z}) \leq a(A)$. Moreover, function $I_A(\cdot)$ is:*

(i) Monotone: $\mathbf{z} \geq \mathbf{z}' \implies I_A(\mathbf{z}) \geq I_A(\mathbf{z}')$.

(ii) Positively homogeneous: $I_A(\alpha \mathbf{z}) = \alpha I_A(\mathbf{z})$ for all $\alpha \geq 0$.

(iii) Concave: $I_A(\lambda \mathbf{z} + (1-\lambda)\mathbf{z}') \geq \lambda I_A(\mathbf{z}) + (1-\lambda) I_A(\mathbf{z}')$ for all $\lambda \in (0,1)$.

(iv) C-additive: $I_A(\mathbf{z} + \beta \cdot \iota) = I_A(\mathbf{z}) + I_A(\beta \cdot \iota)$ for all $\beta \in R$.

(v) Normalized: $I_A(\iota) = 1$.

Proof. Statements (ii), (iv) and (v) hold by construction. Let $f \in Z_{0.2}$ be an act such that $A \in \mathcal{L}(f)$. It exists by the definition of $\mathcal{D}_{0.2}$. Let' prove the main statement $\mathbf{z} \in A \iff I_A(\mathbf{z}) \leq a(A)$, so assume $\mathbf{z} \in Z_{0.4}$.

Case 1. $|\mathbf{z}_\perp| = 0.1$, then $\mathbf{z} \in A \iff I_A(\mathbf{z}) \leq a(A)$ by eq. (13).

Case 2. $|\mathbf{z}_\perp| = 0$, then $\mathbf{z} \in A \iff I_A(\mathbf{z}) \leq a(A)$ by the definition of $a(A)$.

Case 3. $|\mathbf{z}_\perp| \neq 0, 0.1$ and $|\bar{z} - a(A)| \leq |\mathbf{z}_\perp|$. Denote

$$\mathbf{y} \equiv \left[a(A) + 0.1 \frac{\bar{z}}{|\mathbf{z}_\perp|} - 0.1 \frac{a(A)}{|\mathbf{z}_\perp|} \right] \iota + 0.1 \frac{\mathbf{z}_\perp}{|\mathbf{z}_\perp|}$$

Since $a(A) \in [0.4, 0.6]$, we have $0.2\iota = (0.4 - 0.1 - 0.1)\iota \leq \mathbf{y} \leq (0.6 + 0.1 + 0.1)\iota = 0.8\iota$, thus $\mathbf{y} \in Z_{0.6} \subseteq Z_1$. Since $\frac{\mathbf{z}_\perp}{|\mathbf{z}_\perp|} \in O$, by eq. (13) we have:

$$\mathbf{y} \in A \iff \frac{\bar{z}}{|\mathbf{z}_\perp|} - \frac{a(A)}{|\mathbf{z}_\perp|} \leq -J_A\left(\frac{\mathbf{z}_\perp}{|\mathbf{z}_\perp|}\right) \iff I_A\left(\frac{\mathbf{z}}{|\mathbf{z}_\perp|}\right) \leq \frac{a(A)}{|\mathbf{z}_\perp|} \iff I_A(\mathbf{z}) \leq a(A)$$

Case 3.1. $|\mathbf{z}_\perp| < 0.1$. Denote $\lambda = 10|\mathbf{z}_\perp| \in (0,1)$. Then we have

$$\mathbf{z} = \lambda \mathbf{y} + (1-\lambda)a(A) \cdot \iota$$

Let $I_A(\mathbf{z}) \leq a(A)$, then $\{y, a(A) \cdot \iota\} \subset A$, and, since $f \in A$, by *No C-Hedging* $f \in c(A \cup \{\mathbf{z}\})$, thus $\mathbf{z} \in A$ by maximality of $A \in \mathcal{L}(f)$. Now assume that $I_A(\mathbf{z}) > a(A)$. Take $\epsilon = 0.01\min\left\{1, \frac{1-\lambda}{\lambda}, \frac{I_A(\mathbf{z}) - a(A)}{\lambda}\right\} > 0$. Then $\mathbf{y} - \epsilon\iota \in Z_1$, and by eq. (13) and our choice of ϵ also $\mathbf{y} - \epsilon\iota \notin A$. Similarly, $\left(a(A) + \frac{\lambda}{1-\lambda}\epsilon\right)\iota \in Z_1$, and $\left(a(A) + \frac{\lambda}{1-\lambda}\epsilon\right)\iota \notin A$ by definition of $a(A)$. Finally,

$$\mathbf{z} = \lambda(\mathbf{y} - \epsilon\iota) + (1-\lambda)\left(a(A) + \frac{\lambda}{1-\lambda}\epsilon\right)\iota$$

By maximality of $A \in \mathcal{L}(f)$ we get $f \notin c(A \cup \{\mathbf{y} - \epsilon\iota\})$ and $f \notin c\left(A \cup \left\{\left(a(A) + \frac{\lambda}{1-\lambda}\epsilon\right)\iota\right\}\right)$. Therefore, we can apply *Indirect Ambiguity Aversion* to get $f \notin c(A \cup \{\mathbf{z}\})$, implying $\mathbf{z} \notin A$.

Case 3.2: $|\mathbf{z}_\perp| > 0.1$. Denote $\lambda = (10|\mathbf{z}_\perp|)^{-1} \in (0,1)$. Then

$$\mathbf{y} = \lambda \mathbf{z} + (1-\lambda)a(A) \cdot \iota$$

Consider $I_A(\mathbf{z}) > a(A)$. Then $\mathbf{y} \notin A$. Assume for a contradiction that $\mathbf{z} \in A$, then by *No C-Hedging* $f \in c(A \cup \{\mathbf{y}\})$, hence by maximality of A we get $\mathbf{y} \in A$, contradiction. Thus, $\mathbf{z} \notin A$. Consider now $I_A(\mathbf{z}) \leq A$. Then $\mathbf{y} \in A$. Assume for a contradiction that $\mathbf{z} \notin A$. Since $A \cup \{\mathbf{z} - (1/n)\iota\} \to A \cup \{z\}$, then if for arbitrary large n we have $f \in c(A \cup \{\mathbf{z} - (1/n)\iota\})$, the by *Menu Continuity* $f \in c(A \cup \{\mathbf{z}\})$, contradicting maximality of $A \in \mathcal{L}(f)$. Therefore, there exists N such that $\mathbf{z} - (1/n)\iota \notin A$ for all $n \geq N$. Take $\epsilon = 0.01\min\ \{1/N, (1-\lambda)/\lambda\} > 0$. Then $\mathbf{z} - \epsilon \cdot \iota \in Z_1 \backslash A$ and $\left(a(A) + \frac{\lambda}{1-\lambda}\epsilon\right)\iota \in Z_1 \backslash A$. Next,

$$\mathbf{y} = \lambda(\mathbf{z} - \epsilon\iota) + (1-\lambda)\left(a(A) + \frac{\lambda}{1-\lambda}\epsilon\right)\iota$$

Since $f \notin c(A \cup \{\mathbf{z} - \epsilon\iota\})$ and $f \notin c\left(A \cup \left\{\left(a(A) + \frac{\lambda}{1-\lambda}\epsilon\right)\iota\right\}\right)$, the *Indirect Ambiguity Aversion* implies $f \notin c(A \cup \{\mathbf{y}\})$, contradicting $\mathbf{y} \in A$.

Case 4. $|\mathbf{z}_\perp| \neq 0, 0.1$ and $\overline{z} - a(A) > |\mathbf{z}_\perp|$. Then $\mathbf{z} = \overline{z}\iota + \mathbf{z}_\perp >> (a(A) + 0.5(\overline{z} - a(A) - |\mathbf{z}_\perp|))\iota >> a(A)\iota$. Thus $f \notin c(A \cup \{(a(A) + 0.5(\overline{z} - a(A) - |\mathbf{z}_\perp|))\iota\})$ by the definition of $a(A)$, and by *Monotonicity* $f \notin c(A \cup \{\mathbf{z}\})$, implying $\mathbf{z} \notin A$. In this case $I_A(\mathbf{z}) \geq \overline{z} - |\mathbf{z}_\perp| > a(A)$, so the main statement of the Lemma holds.

Case 5. $|\mathbf{z}_\perp| \neq 0, 0.1$ and $\overline{z} - a(A) < -|\mathbf{z}_\perp|$. Then $\mathbf{z} = \overline{z}\iota + \mathbf{z}_\perp \leq a(A)\iota \in A$. Hence, $f \in c(A \cup \{\mathbf{z}\})$ by *Monotonicity*, and $\mathbf{z} \in A$ by maximality of A. From the other hand, in this case $I_A(\mathbf{z}) \leq \overline{z} + |\mathbf{z}_\perp| \leq a(A)$, so the main statement of the Lemma holds.
Case 1-5 cover all possibilities, thus $\mathbf{z} \in A \Longleftrightarrow I_A(\mathbf{z}) \leq a(A)$ for all $\mathbf{z} \in Z_{0.4}$, $A \in \mathcal{D}_{0.2}$.

Proof of (iii). It is easy to see from the definition of $I_A(\cdot)$ that $|I_A(\mathbf{z})| \leq 2|\mathbf{z}|$ for any $\mathbf{z} \in R^{|S|}$. Consider arbitrary $\mathbf{z}, \mathbf{z}' \in R^{|S|}$. If $\mathbf{z} = \mathbf{z}' = \mathbf{0}$, then $I_A(\mathbf{z}+\mathbf{z}') = I_A(0) = 0 = I_A(0) + I_A(0) = I_A(\mathbf{z}) + I_A(\mathbf{z}')$. Otherwise, denote $t = 0.01 \cdot (\max\{|z|, |z'|\})^{-1} > 0$. Then $-0.02 \leq I_A(t\mathbf{z}), I_A(t\mathbf{z}') \leq 0.02$. Consider

$$\tilde{\mathbf{z}} \;=\; t\mathbf{z} + (a(A) - I_A(t\mathbf{z}))\,\iota \quad , \quad \tilde{\mathbf{z}'} \;=\; t\mathbf{z}' + (a(A) - I_A(t\mathbf{z}'))\,\iota$$

Take arbitrary $\epsilon \in (0, 0.01)$. Since $a(A) \in [0.4, 0.6]$, then $0.3\iota \leq \tilde{\mathbf{z}} + \epsilon\iota, \tilde{\mathbf{z}'} + \epsilon\iota \leq 0.7\iota$, hence $\tilde{\mathbf{z}} + \epsilon\iota, \tilde{\mathbf{z}'} + \epsilon\iota \in Z_{0.4}$, thus the main statement of the Lemma applies to these acts. Next, $I_A(\tilde{\mathbf{z}} + \epsilon\iota) = I_A(\tilde{\mathbf{z}'} + \epsilon\iota) = a(A) + \epsilon > a(A)$, hence $\tilde{\mathbf{z}} + \epsilon\iota, \tilde{\mathbf{z}'} + \epsilon\iota \notin A$. Thus, $f \notin c(A \cup \{\tilde{\mathbf{z}} + \epsilon\iota\})$ and $f \notin c(A \cup \{\tilde{\mathbf{z}}' + \epsilon\iota\})$, hence by *Indirect Ambiguity Aversion*, maximality of A and the fact that $Z_{0.4}$ is convex, for all $\lambda \in (0,1)$ we have:

$$\lambda(\tilde{\mathbf{z}} + \epsilon\iota) + (1-\lambda)(\tilde{\mathbf{z}'} + \epsilon\iota) \;\in Z_{0.4} \backslash A \;\Longrightarrow\; I_A(\lambda(\tilde{\mathbf{z}} + \epsilon\iota) + (1-\lambda)(\tilde{\mathbf{z}'} + \epsilon\iota)) \;>\; a(A) \;\Longrightarrow$$

$$\Longrightarrow\; I_A(\lambda\tilde{\mathbf{z}} + (1-\lambda)\tilde{\mathbf{z}'}) + \epsilon \;>\; a(A) \;\Longrightarrow\; I_A(\lambda\tilde{\mathbf{z}} + (1-\lambda)\tilde{\mathbf{z}'}) \;\geq\; a(A) \;\Longrightarrow$$

$$\Longrightarrow\; I_A\left(\lambda\left[t\mathbf{z} + (a(A) - I_A(t\mathbf{z}))\,\iota\right] + (1-\lambda)\left[t\mathbf{z}' + (a(A) - I_A(t\mathbf{z}'))\,\iota\right]\right) \;\geq\; a(A) \;\Longrightarrow$$

$$\Longrightarrow\; I_A(\lambda t\mathbf{z} + (1-\lambda)t\mathbf{z}') \;\geq\; \lambda I_A(t\mathbf{z}) + (1-\lambda) I_A(t\mathbf{z}') \;\Longrightarrow$$

$$\Longrightarrow\; I_A(\lambda\mathbf{z} + (1-\lambda)\mathbf{z}') \;\geq\; \lambda I_A(\mathbf{z}) + (1-\lambda) I_A(\mathbf{z}')$$

Thus, $I_A(\cdot)$ is a concave function.

Proof of (i). Consider arbitrary $\mathbf{z}, \mathbf{z}' \in R^{|S|}$ and let $\mathbf{z} \geq \mathbf{z}'$. If $\mathbf{z} = \mathbf{z}' = \mathbf{0}$, then the statement holds. Otherwise denote $t = 0.01 \cdot (\max\{|z|, |z'|\})^{-1} > 0$ and

$$\widetilde{\mathbf{z}} \;=\; t\mathbf{z} + (a(A) - I_A(t\mathbf{z}))\,\iota \quad , \quad \widetilde{\mathbf{z}'} \;=\; t\mathbf{z}' + (a(A) - I_A(t\mathbf{z})\,\iota$$

Notice that we use $I_A(t\mathbf{z})$ and not $I_A(t\mathbf{z}')$ in the definition of $\widetilde{\mathbf{z}'}$, like we did to prove concavity of $I_A(\cdot)$. This way $I_A(\widetilde{\mathbf{z}}) = a(A)$, $\widetilde{\mathbf{z}}' \leq \widetilde{\mathbf{z}}$, and $\widetilde{\mathbf{z}}, \widetilde{\mathbf{z}}' \in Z_{0.4}$. Therefore, $\widetilde{\mathbf{z}} \in A$, and by *Monotonicity* $f \in c(A \cup \{\widetilde{\mathbf{z}'}\})$, thus by maximality of A we get $\widetilde{\mathbf{z}'} \in A$, hence $I_A(\widetilde{\mathbf{z}}') \leq a(A) = I_A(\widetilde{\mathbf{z}})$. Using statements (ii),(iv) and (v), we get $I_A(\mathbf{z}') \leq I_A(\mathbf{z})$. ■

Lemma 8. *Let $\mathcal{I}$ be the set of monotone positively homogeneous concave constant additive and normalized functions $I: R^{|S|} \to R$, and Π be the set of non-empty convex and closed sets of probabilities $P \subseteq \triangle S$. Then there is a bijection $\tau: \Pi \to \mathcal{I}$ given by*

$$\tau(P)(\mathbf{z}) \equiv \min_{\mu \in P} \sum_{s \in S} \mu(s) z_s \qquad \forall \mathbf{z} \in R^{|S|} \tag{16}$$

$$\tau^{-1}(I) = \sigma(I) \equiv \left\{ \mu \in \triangle S \,\middle|\, \sum_{s \in S} \mu(s) z_s \geq I(\mathbf{z}) \; \forall \mathbf{z} \in R^{|S|} \right\} \tag{17}$$

Proof. This is a well-known statement. We provide a full proof in Appendix C. ■

Recall the definition of the set of menus $\mathcal{D}_{0.2}$ given by eq. (10) and function I_A given by eq. (15). Define a collection of sets of beliefs $\mathcal{A}$ by

$$\mathcal{A} = \{P \in \Pi \mid \exists A \in \mathcal{D}_{0.2}: \; I_A = \tau(P)\} \tag{18}$$

Notice that $\mathcal{A} \neq \emptyset$, since $0.5\iota \in Z_{0.2}$, hence there exists $A \in \mathcal{L}(0.5\iota) \subseteq \mathcal{D}_{0.2}$, and $I_A \in \mathcal{I}$, thus $\tau(P) = I_A$ for $P = \sigma(I_A)$. Next, by lemma 3 $U(\cdot)$ is non-degenerate. Therefore, $(U, cl(\mathcal{A}))$ is a well-defined framed ambiguity model, where $cl(\mathcal{A})$ is a closure of $\mathcal{A}$ with respect to the Hausdorff metric on Π. Let us abuse notation and say that model $(U, \mathcal{A})$, where U is a vNM expected utility function, and $\emptyset \neq \mathcal{A} \subseteq \Pi$ represents a choice correspondence $c(\cdot)$ if $c(A)$ is given by eq. (1) but $\mathcal{A}$ is not necessary closed.

Lemma 9. *Let U be a vNM expected utility function that represents c_0 and $\mathcal{A}$ be defined via eq. (18). Then model $(U, \mathcal{A})$ represents $c(\cdot)$ on the set of menus consisting of canonical acts $\widehat{f} \in H = Z_1 = [0,1]^{|S|}$.*

Proof. Let $c_{(U,\mathcal{A})}$ be choice correspondence induced by the model $(U, \mathcal{A})$, given by eq. (1). By definition $\mathbf{z} \in c_{(U,\mathcal{A})}(B)$ if and only if there exists $P \in \mathcal{A}$ such that $W_P(\mathbf{z}) \geq W_P(\mathbf{z}')$ for all $\mathbf{z}' \in B$, where we implicitly use $U(\cdot)$ in the definition of $W_P(\cdot)$. Lemma 8 implies $\forall B \in \mathcal{K}$

$$\mathbf{z} \in c_{(U,\mathcal{A})}(B) \iff \exists A \in \mathcal{D}_{0.2}: \; I_A(\mathbf{z}) \geq I_A(\mathbf{z}') \; \forall \mathbf{z}' \in B$$

Step 1. We'll firstly show that the statement is true for all menus $B \subseteq Z_{0.2}$.

Step 1.1. We want to show that $\mathbf{z} \in Z_{0.2}$ and $A \in \mathcal{L}(\mathbf{z})$ imply $I_A(\mathbf{z}) = a(A)$. By lemma 7 $I_A(\mathbf{z}) \leq a(A)$. Assume for a contradiction that $I_A(\mathbf{z}) < a(A)$. Consider $\mathbf{z}' = \mathbf{z} + 0.5 \cdot \min\{0.01, a(A) - I_A(\mathbf{z})\}\iota$. Then $\mathbf{z}' \in Z_{0.4}$ and also $I_A(\mathbf{z}') \leq 0.5 I_A(\mathbf{z}) + 0.5a(A) < a(A)$, hence $\mathbf{z}' \in A$. Since $\mathbf{z} \in c(A)$, α implies $\mathbf{z} \in c(\{\mathbf{z}, \mathbf{z}'\})$, but $\mathbf{z}' >> \mathbf{z}$, contradicting *Strict Monotonicity*.

Step 1.2. We want to show that $c(B) \subseteq c_{(U,\mathcal{A})}(B)$. Let $\mathbf{z} \in c(B)$. Then there is $A \in \mathcal{L}(\mathbf{z})$ such that $B \subseteq A$ and $\mathbf{z} \in c(A)$. Using Lemma 7, Lemma 8 and Step 1.1, we get $I_A(\mathbf{z}') \leq a(A) = I_A(\mathbf{z})$ for all $z' \in B \subseteq A$. Therefore, $W_{\sigma(I_A)}(\mathbf{z}) \geq W_{\sigma(I_A)}(\mathbf{z}')$ for all $\mathbf{z}' \in B$, hence $\mathbf{z} \in c_{(U,\mathcal{A})}(B)$.

Step 1.3. We want to show that $c_{(U,\mathcal{A})}(B) \subseteq c(B)$. Let $\mathbf{z} \in c_{(U,\mathcal{A})}(B)$. Then there is $f \in Z_{0.2}$ and $A \in \mathcal{L}(f)$ such that $I_A(\mathbf{z}) \geq I_A(\mathbf{z}')$ for all $\mathbf{z}' \in B$.

Case 1: $I_A(\mathbf{z}) = a(A)$. Then for all $\mathbf{z}' \in B$ we have $I_A(\mathbf{z}') \leq a(A)$, hence by Lemma A.6 $\mathbf{z}' \in A$, thus $B \subseteq A$. Consider $\mathbf{z}^n = \mathbf{z} + (1/100n)\iota \in Z_{0.4}$ for $n = 1, 2, \ldots$. Notice that $\mathbf{z}^n \longrightarrow \mathbf{z}$. Next, $I_A(\mathbf{z}^n) = I_A(\mathbf{z}) + (1/100n) = a(A) + (1/100n)$, therefore by Lemma A.6 $\mathbf{z}^n \notin A$. By *Aizerman's Property* $\mathbf{z}^n \in c(A \cup \{\mathbf{z}^n\})$. By *Act Continuity* $\mathbf{z} \in c(A \cup \{\mathbf{z}\}) = c(A)$, and α implies $\mathbf{z} \in c(B)$.

Case 2: $I_A(\mathbf{z}) > a(A)$. Denote $\lambda = \dfrac{0.7 - a(A)}{0.7 - I_A(\mathbf{z})}$. Observe that $\lambda \in (0,1)$, since $a(A) \in [0.4, 0.6]$. Then for all $\mathbf{z}' \in B$ we have

$$I_A(\lambda \mathbf{z}' + (1-\lambda)(0.7\iota)) \;=\; \lambda I_A(\mathbf{z}') + (1-\lambda) \cdot 0.7 \;\leq$$

$$\leq\; \lambda I_A(\mathbf{z}) + (1-\lambda) \cdot 0.7 \;=\; a(A) = I_A(\lambda \mathbf{z} + (1-\lambda)(0.7\iota))$$

We can apply Case 1 to conclude that $\lambda \mathbf{z} + (1-\lambda)(0.7\iota) \in c(\lambda B + (1-\lambda)\{0.7\iota\})$. Then by *C-Independence* $\mathbf{z} \in c(B)$.

Case 3: $I_A(\mathbf{z}) > a(A)$. Take $\lambda = \dfrac{a(A) - 0.3}{I_A(\mathbf{z}) - 0.3} \in (0,1)$, and then repeat the proof used for Case 2 with menu $\lambda B + (1-\lambda)\{0.3\iota\}$ to get $\mathbf{z} \in c(B)$.

Step 2. We finally show that the statement is true for all $B \in \mathcal{K}$

$$\mathbf{z} \in c_{(U,\mathcal{A})}(B) \iff \exists f \in Z_{0.2} : \exists A \in \mathcal{L}(f) : \; I_A(\mathbf{z}) \geq I_A(\mathbf{z}') \; \forall \mathbf{z}' \in B \iff$$

$$\iff \exists f \in Z_{0.2} : \exists A \in \mathcal{L}(f) : \; I_A(0.2\mathbf{z} + 0.8 \cdot (0.5\iota)) \geq I_A(0.2\mathbf{z}' + 0.8 \cdot (0.5\iota)) \; \forall \mathbf{z}' \in B \iff$$

$$\iff 0.2\mathbf{z} + 0.8 \cdot (0.5\iota) \in c_{(U,\mathcal{A})}(0.2B + 0.8 \cdot (0.5\iota)) \iff$$

$$\iff 0.2\mathbf{z} + 0.8 \cdot (0.5\iota) \in c(0.2B + 0.8 \cdot (0.5\iota)) \iff \mathbf{z} \in c(B)$$

where we used properties of $I_A(\cdot)$ given by Lemma 7, the fact that $0.2B + 0.8 \cdot (0.5\iota) \subseteq Z_{0.2}$, Step 1 and *C-Independence*. ■

Lemma 10. *Model $(U, \mathcal{A})$ represents a choice correspondence $c(\cdot)$ if and only if it represents choice correspondence $c(\cdot)$ restricted on the set of menus, consisting of canonical acts.*

Proof. We have already established that $f \sim g$ is equivalent to $U(f(s)) = U(g(s))$ for all $s \in S$, which implies $W_P(f) = W_P(\widehat{f})$ for all $f \in H$, $P \in \mathcal{A}$. Therefore, $\widehat{f} \in c_{(U,\mathcal{A})}(\widehat{B}) \iff f \in c_{(U,\mathcal{A})}(B)$ for a canonical act $\widehat{f} \sim f$ and canonical menu $\widehat{B}$. From the other hand, lemma 5 says that $f \in c(B) \iff \widehat{f} \in c(\widehat{B})$. ■

Lemmas 9 and 10 show that $c(\cdot)$ admits a representation given by eq. (1) with non-degenerate $U(\cdot)$ (by lemma 3), but we don't know if $\mathcal{A}$ is closed.

Lemma 11. *If axioms 1-7 hold for $c(\cdot)$, and model $(U, \mathcal{A})$ represents $c(\cdot)$, then model $(U, cl(\mathcal{A}))$ represent $c(\cdot)$ as well.*

Proof. Let $P^k \in \mathcal{A}$ for all $k = 1, 2, ...$, and $P^k \longrightarrow P \in \Pi$. Notice that $(U, \{P\})$ is a framed ambiguity model, and let $c_{\{P\}}(\cdot)$ be the choice correspondence represented by this model. By the proven first part of the Theorem, $c_{\{P\}}(\cdot)$ satisfies Axioms 1-7. Consider an arbitrary menu $A \in \mathcal{K}$, and let $f \in c_{\{P\}}(A)$. Then by *C-Independence* we get $f_{0.5} \in c_{\{P\}}(A_{0.5})$, where $f_{0.5} = 0.5f + 0.5(0.5x + 0.5y)$ and $A_{0.5} = 0.5A + 0.5(0.5x + 0.5y)$. Therefore, $W_P(f_{0.5}) \geq W_P(g_{0.5})$ for all $g \in A$. Take $f^n_{0.5} = (1 - 1/n)f_{0.5} + (1/n)x$. Then $W_P(f^n_{0.5}) - W_P(g_{0.5}) \geq 1/(4n)$ for all $g \in A$. By lemma 1, $W_P(h)$ is continuous w.r.t. (P, h), thus for large enough $N(k)$ we have $W_{P^k}(f^n_{0.5}) \geq W_{P^k}(g)$ for all $n \geq N(k)$ and all $g \in A$. Therefore, $f^n_{0.5} \in c(A_{0.5} \cup \{f^n_{0.5}\})$ for all $n \geq N(k)$. Take $n(k) = \max\{N(k), k\}$, then $f^{n(k)}_{0.5} \longrightarrow f_{0.5}$, hence by *Menu Continuity* $f_{0.5} \in c(A_{0.5} \cup \{f_{0.5}\}) = c(A_{0.5})$. *C-Independence* implies $f \in c(A)$. Thus $c_{\{P\}}(A) \subseteq c(A)$ for all $A \in \mathcal{K}$, and we conclude that $c'(A) \subseteq c(A)$, where $c'(\cdot)$ is a choice correspondence, represented by the framed ambiguity model $(U, \mathcal{A})$. From the other hand, since model $(U, \mathcal{A})$ represents $c(\cdot)$ and $\mathcal{A} \subseteq cl(\mathcal{A})$, then $c(\cdot) \subseteq c'(\cdot)$. We conclude that $c'(\cdot) = c(\cdot)$, thus framed ambiguity model $(U, cl(\mathcal{A}))$ represents $c(\cdot)$. ■

Part 2 of Theorem 1 follows from lemmas 9, 10 and 11. ■

Proof of Theorem 2

Step 1. Let $(U, cl(\mathcal{A}))$ be a framed ambiguity representation of $c(\cdot)$ constructed in the proof of Theorem 1, and $(U', \mathcal{A}')$ be an arbitrary framed ambiguity representation. Since both U and U' represent the same restriction of choice correspondence on menus consisting of constant acts, then U' is a positive affine transformation of U by Lemma 3. Since for a maxmin utility function $W_P(\cdot)$ for any $P \in \Pi$ and any acts $f, g \in H$ statement $W_P(f) \geq W_P(g)$ is invariant with respect to the positive affine transformations of the associated vNM expected utility function, we can normalize $U' = U$.

Step 2. By lemma 10 we may assume $H = [0,1]^{|S|}$. Recall the definitions of $\mathcal{L}(\cdot)$, $a(\cdot)$, $I_A(\cdot)$, $\tau(\cdot)$, $\sigma(\cdot)$, $\mathcal{A}$ given by eqs. (7), (9), (15), (16), (17), (18), and properties of function $I_A(\cdot)$ given by Lemma 7. Let $P \in \mathcal{A}$. Then $\exists f \in Z_{0.2}$ and $A \in \mathcal{L}(f)$ such that $I_A = \tau(P)$ and $A = \{\widetilde{\mathbf{z}} \mid I_A(\widetilde{\mathbf{z}}) \leq a(A)\}$. Moreover, $I_A(f) = a(A)$ by Step 1.1. of Lemma 9, and $a(A) \in [0.4, 0.6]$. Let

$$\mathcal{C}' \;=\; \{\, P' \in \mathcal{A}' \mid A \subseteq B(P') \,\} \qquad , \qquad \text{where } \; B(P') = \{\, \widetilde{\mathbf{z}} \in H \mid \tau(P')(f) \geq \tau(P')(\widetilde{\mathbf{z}}) \,\}$$

Since $f \in c(A)$, and $(U, \mathcal{A}')$ represents $c(\cdot)$, then there is some $P' \in \mathcal{A}'$ such that $W_{P'}(f) \geq W_{P'}(\widetilde{\mathbf{z}})$ for all $\widetilde{\mathbf{z}} \in A$, where we use function $U(\cdot)$ in the definition of $W_{P'}(\cdot)$ given by eq. (2). Thus, $W_{P'}(\widetilde{\mathbf{z}}) = \tau(P')(\widetilde{\mathbf{z}})$ for all $\widetilde{\mathbf{z}} \in H$. We conclude that $\mathcal{C}' \neq \varnothing$.

Step 3. Assume for a contradiction that $B(P') \not\subseteq A$ for some $P' \in \mathcal{C}'$. Then by Lemma 6 there exists $D \in \mathcal{L}(f)$ such that $A \subset B(P') \subseteq D$ and $A \neq D$, contradicting the maximality of $A \in \mathcal{L}(f)$. We conclude that $A = B(P')$ for all $P' \in \mathcal{C}'$. In particular, $a(A) = I_A(f) = \tau(P')(f)$, since $b\iota \in A \iff b\iota \in B(P')$, and $I_A(b\iota) = b = \tau(P')(b\iota)$ for all $b \in [0,1]$.

Step 4. Let us show that $I_A(\widetilde{\mathbf{z}}) = \tau(P')(\widetilde{\mathbf{z}})$ for all $\widetilde{\mathbf{z}} \in R^{|S|}$ for all $P' \in \mathcal{C}'$. If $\widetilde{\mathbf{z}} = \mathbf{0}$, then $I_A(\mathbf{0}) = 0 = \tau(P')(\mathbf{0})$. Otherwise consider $\mathbf{w} = 0.1\frac{\widetilde{\mathbf{z}}}{|\widetilde{\mathbf{z}}|} + 0.5\iota \in Z_{0.2}$. Then $\mathbf{w} + 0.4b\iota \in Z_1 = H$ for all $b \in [-1,1]$. Notice also that $\mathbf{w} - 0.4\iota << 0.4\iota \leq a(A)\iota \leq 0.6\iota << \mathbf{w} + 0.4\iota$. Therefore, since I_A and $\tau(P')$ are monotone, $I_A(\mathbf{w} - 0.4\iota), \tau(P')(\mathbf{w} - 0.4\iota) < a(A) < I_A(\mathbf{w} - 0.4\iota), \tau(P')(\mathbf{w} - 0.4\iota)$. Next, by C-additivity $I_A(\mathbf{w} + 0.4b\iota) = I_A(\mathbf{w}) + 0.4b$ and $\tau(P')(\mathbf{z} + 0.4b\iota) = \tau(P')(\mathbf{w}) + 0.4b$. Finally, by Step 3 $I_A(\mathbf{w} + 0.4b\iota) \geq a(A) \iff \tau(P')(\mathbf{w} + 0.4b\iota) \geq a(A)$ for all $b \in [-1,1]$. Therefore, there is $b \in [-1,1]$ such that $I_A(\mathbf{w} + 0.4b\iota) = a(A) = \tau(P')(\mathbf{w} + 0.4\iota)$. By C-additivity and positive homogeneity of I_A and $\tau(P')$ we get $I_A(\mathbf{w}) = \tau(P')(\mathbf{w})$ and $I_A(\widetilde{\mathbf{z}}) = \tau(P')(\widetilde{\mathbf{z}})$. Since this holds for arbitrary $\widetilde{\mathbf{z}}$, then $I_A = \tau(P')$. Hence, $P = \sigma(\tau(P)) = \sigma(I_A) = \sigma(\tau(P')) = P'$. We conclude that $P \in \{P\} = \mathcal{C}' \subseteq \mathcal{A}'$. Since it holds for all $P \in \mathcal{A}$, we get $\mathcal{A} \subseteq \mathcal{A}'$. Therefore, $cl(\mathcal{A}) \subseteq cl(\mathcal{A}') = \mathcal{A}'$. We conclude that $(U, cl(\mathcal{A}))$ is a minimum framed ambiguity representation of $c(\cdot)$.

Step 5. Consider

$$\mathcal{B} \quad = \bigcup_{\mathcal{A}' \subseteq \Pi:\ (U,\mathcal{A}') \text{ is a framed ambiguity model representing } c(\cdot)} \mathcal{A}' \tag{19}$$

We show that model $(U, \mathcal{B})$ represents $c(\cdot)$. Let act f maximize $W_P(\cdot)$ over menu B for some $P \in \mathcal{B}$, where we use $U(\cdot)$ in the definition of $W_P(\cdot)$ according to eq. (1). Then there is a framed ambiguity representation $(U, \mathcal{A}')$ of $c(\cdot)$ such that $P \in \mathcal{A}'$, hence $f \in c(B)$. For the converse, let $f \in c(B)$. Take arbitrary representation $(U, \mathcal{A}')$, then there is $P \in \mathcal{A}' \subseteq \mathcal{B}$ such that f maximizes W_P over menu B. Thus indeed $(U, \mathcal{B})$ represents $c(\cdot)$. Finally, for any framed ambiguity representation $(U, \mathcal{A}')$ we get $\mathcal{A}' \subseteq \mathcal{B}$. By lemma 11 a framed ambiguity model $(U, cl(\mathcal{B}))$ represents $c(\cdot)$, thus $\mathcal{B} = cl(\mathcal{B})$, and we conclude that $(U, \mathcal{B})$ is a maximum framed ambiguity representation of $c(\cdot)$.

Step 6. Finally, let $cl(\mathcal{A}) \subseteq \mathcal{A}' \subseteq \mathcal{B}$, and $\mathcal{A}'$ be closed. Then $c'(\cdot) \subseteq c''(\cdot) \subseteq c'''(\cdot)$, where $c'(\cdot)$, $c''(\cdot)$ and $c'''(\cdot)$ are choice correspondences represented by framed ambiguity models $(U, cl(\mathcal{A}))$, $(U, \mathcal{A}')$ and $(U, \mathcal{B})$. Since by previous steps $c'(\cdot) = c'''(\cdot) = c(\cdot)$, then $c''(\cdot) = c(\cdot)$, hence $(U, \mathcal{A}')$ represents $c(\cdot)$. Theorem 2 is proven. ■

Proof of Proposition 2

Consider the following alternative definition of the coherent intersection:

DEFINITION 4*: Given a non-empty collection of sets of beliefs $\mathcal{C} \subseteq \Pi$, a set of beliefs $P \in \Pi$ is a *coherent-* intersection* of $\mathcal{C}$ if its support functional is equal to the maximum of the support functionals of sets of beliefs in $\mathcal{C}$, and this maximum exists:

$$I_P(z) = \max_{P' \in \mathcal{C}} I_{P'}(z) \qquad \forall z \in R^{|S|} \tag{20}$$

$$\text{where } \ I: \ R^{|S|} \to R \quad \text{is given by } \ I_P(z) \ = \ \min_{\mu \in P} \sum_{s \in S} \mu(s) \cdot z_s$$

Consider a natural embedding of $\triangle S$ in R^n. Lemma 12 proves statement (i) of the Proposition.

Lemma 12. *There exists a coherent-* intersection P^* of a non-empty collection of sets of beliefs $\mathcal{C} \subseteq \Pi$ if and only if there exists a coherent intersection P of $\mathcal{C}$, where sets of beliefs are considered as subsets of R^n with the requirement that $\sum_{i=1}^n \mu(i) = 1$; moreover, $P^* = P$.*

Proof. Suppose that P is a coherent intersection of $\mathcal{C}$. For an arbitrary $z \in R^{|S|}$, let z' be its component, orthogonal to $(1,...,1)$. The projections of P and $\mathcal{C}$ on the linear subspace $R_{z'} = \{x \in R^n \mid \exists \lambda \in R : \; x = \lambda z'\}$ are an interval $[a(P), b(P)]$ and a collection of intervals $\{[a(P'), b(P')]\}_{P' \in \mathcal{C}}$. Then, without loss of generality, $I_{P'}(z) = a(P')$ for all $P' \in \mathcal{C} \cup P$. Since P is a coherent intersection, then

$$[a(P), b(P)] \;=\; \bigcap_{P' \in \mathcal{C}} [a(P'), b(P')]$$

It follows that

$$I_P(z) - \sum_{i=1}^{S} z_s \;=\; I_P(z') \;=\; a(P) \;=\; \max_{P' \in \mathcal{C}} a(P') \;=\; \max_{P' \in \mathcal{C}} I(P')(z') \;=\; \max_{P' \in \mathcal{C}} I_{P'}(z') - \sum_{i=1}^{S} z_s$$

Thus, P is a coherent-* intersection of $\mathcal{C}$.

Claim 3. *Let $P \subseteq R^n$ be the intersection of sets in a collection $\mathcal{C}$, P_o and $\mathcal{C}_o$ are projections of P and $\mathcal{C}$ on an arbitrary linear subspace R_o and let $\tilde{P}$ be an intersection of sets in $\mathcal{C}_o$, then $P_o \subseteq \tilde{P}$.*

Proof of the claim. Consider arbitrary $\hat{x} \in P_o$, then there is $x \in P$ such that $x'' = x_o$, where x_o is a projection of x on R_o. It follows that $x \in P'$ for all $P' \in \mathcal{C}$, hence $x'' = x_o \in P'_o$ for all $P' \in \mathcal{C}$, hence $x'' \in \tilde{P}$. The Claim is proven. ■

Suppose now that P^* is a coherent-* intersection of $\mathcal{C}$. Let P^*_o and $\mathcal{C}_o$ be projections of P and $\mathcal{C}$ on an arbitrary linear subspace R_o of R^n that can be shifted to a subspace of $\triangle S$; that is, $\sum_{i=1}^n x_i = const$ for all $x \in R_o$. Towards a contradiction, assume that P^*_o is not the intersection of sets in $\mathcal{C}_o$. By Claim 3,

$$\bigcap_{P'_o \in \mathcal{C}_o} P'_o \supsetneq P^*_o$$

Since all sets are convex, by the Separating Hyperplane Theorem, the above equation implies that there exists $z \in R^s$ such that

$$I_{\bigcap_{P'_o \in \mathcal{C}_o} P'_o}(z) \;<\; I_{P^*_o}(z)$$

It follows that

$$\max_{P'_o \in \mathcal{C}_o} I_{P'_o}(z) \ \le \ I_{\bigcap_{P'_o \in \mathcal{C}_o} P'_o}(z) \ < \ I_{P^*_o}(z)$$

Let $z_\perp$ be a component of z that is orthogonal to R_o, and $z' = z - z_\perp$. Since $I_{P'_o}(z_\perp) = I_{P'_o}(z_\perp)$ for any $P'_o, P''_o \in \mathcal{C}_o \cup \{P_o\}$, then

$$\max_{P'_o \in \mathcal{C}_o} \ I_{P'_o}(z') \ < \ I_{P^*_o}(z')$$

and

$$\max_{P' \in \mathcal{C}} I_{P'}(z') \ < \ I_{P^*}(z')$$

in contradiction to P^* being a coherent-* intersection of $\mathcal{C}$. For a linear subspace R_o that cannot be shifted to a subspace of $\triangle S$, consider the additional projection to $\{x \in R_o | \sum_{i=1}^n x_i = 1\}$, and repeat the argument above. ■

Consider a binary state space $|S| = 2$ and a non-empty closed collection $\mathcal{C} \subseteq \Pi$. Then $\triangle S$ is isomorphic to $[0,1] \subset R$, where $\mu \in [0,1]$ corresponds to $\mu(1)$, and $\mu(2) = 1 - \mu(1)$. Thus we can consider each set of beliefs as an interval $P' = [a', b'] \subseteq [0,1]$ with $a' \le b'$. Assume that there is a non-empty intersection $\overline{P}$ of sets of beliefs in $\mathcal{C}$. Then

$$\overline{P} \ = \ \bigcap_{P' \in \mathcal{C}} P' \ = \ \bigcap_{[a',b'] \in \mathcal{C}} [a', b'] \ = \ \left[\sup_{[a',b'] \in \mathcal{C}} a' \ , \ \inf_{[a',b'] \in \mathcal{C}} b' \right] \ = \ \left[\max_{[a',b'] \in \mathcal{C}} a' \ , \ \min_{[a',b'] \in \mathcal{C}} b' \right] \ \ne \ \emptyset$$

where we used closedness of $\mathcal{C}$ to get maximum and minimum instead of suprenum and infinum. Consider $\mathbf{z} = (z_1, z_2) \in R^2$ with $z_1 \ge z_2$. Then

$$I_{\overline{P}}(z_1, z_2) = \left(\max_{[a',b'] \in \mathcal{C}} a' \right) z_1 + \left(1 - \left(\max_{[a',b'] \in \mathcal{C}} a' \right) \right) z_2 = \max_{[a',b'] \in \mathcal{C}} I_{[a',b']}(z_1, z_2)$$

A symmetric argument works when $z_1 < z_2$. Thus $\overline{P}$ is a coherent intersection of $\mathcal{C}$, proving (ii).

Consider now $|S| > 2$. Let $\mu_1 = (0.1, 0.1, 0.8, ...)$, $\mu_2 = (0.3, 0.1, 0.6, ...)$, $\mu_3 = (0.2, 0.2, 0.6, ...)$, where ... denotes zeros for all s except of the first 3 states. Let $\mathcal{C} = \{P_1, P_2\}$ with $P_1 = conv(\{\mu_1, \mu_3\})$ and $P_2 = conv(\{\mu_2, \mu_3\})$. Then $P_1 \cap P_2 = \{\mu_3\}$. From the other hand, consider $\mathbf{z} \in R^{|S|}$ with $z_2 = 1$ and $z_i = 0$ for all $i \ne 2$. Then $I_{P_1}(\mathbf{z}) = I_{P_2}(\mathbf{z}) = 0.1 < 0.2 = I_{\{\mu_3\}}$. Thus, $\mathcal{C} \subset \Pi$ is a closed collection of sets of beliefs with non-empty intersection, but it fails to have a coherent intersection, otherwise it should have been $\mu_3 = P_1 \cap P_2$ by Step 2, but this is not the case. Proposition 2 is proven. ■

Proof of Proposition 3 and Corollary 1

Step 1. For a fixed vNM expected utility function $U(\cdot)$ and some non-empty collection of sets of beliefs $\mathcal{C} \subseteq \Pi$ denote $c_{\mathcal{C}}(\cdot)$ to be a choice correspondence that is represented by $(U, \mathcal{C})$. Such choice correspondences exist as we have argued in Section 2.3. Notice that by eq. (1) $\forall B \in \mathcal{K}$

$$c_{\mathcal{C}\cup\mathcal{D}}(B) \;=\; c_{\mathcal{C}}(B) \cup c_{\mathcal{D}}(B) \quad , \quad \mathcal{C} \subseteq \mathcal{A} \implies c_{\mathcal{C}}(B) \subseteq c_{\mathcal{A}}(B) \quad , \tag{21}$$

$$c_{\mathcal{C}}(B) \;=\; \bigcup_{P\in\mathcal{C}} c_{\{P\}}(B)$$

Let P be a non-trivial coherent intersection of a non-empty subcollection of sets of beliefs $\mathcal{C} \subseteq \mathcal{A}$. Then for any $f \in H$ and $B \in \mathcal{K}$ we have

$$\Big[f \in c_{\{P\}}(B) \implies W_P(f) \geq W_P(g)\ \forall g \in B \implies \max_{P' \in \mathcal{C}} W_{P'}(f) \;\geq\; \max_{P' \in \mathcal{C}} W_{P'}(g)\ \forall g \in B \implies$$

$$\exists P'' \in \mathcal{C}: \;\; W_{P''}(f) \;\geq\; \max_{P' \in \mathcal{C}} W_{P'}(g) \;\geq\; W_{P''}(g)\ \forall g \in B \implies f \in c_{\mathcal{C}}(B) \Big] \implies c_{\{P\}}(B) \subseteq c_{\mathcal{C}}(B)$$

Since P is a non-trivial coherent intersection of $\mathcal{C}$, then $\{P\} \notin \mathcal{C}$. Therefore, $\mathcal{C} \subseteq \mathcal{A}\backslash\{P\}$, and for any $B \in \mathcal{K}$ we get

$$c_{\mathcal{A}\cup\{P\}}(B) \;=\; c_{\mathcal{A}\backslash\{P\}}(B) \cup c_{\{P\}}(B) \;\subseteq\; c_{\mathcal{A}\backslash\{P\}}(B) \cup c_{\mathcal{C}}(B) \;=\; c_{\mathcal{A}\backslash\{P\}}(B)$$

Therefore, $c_{\mathcal{A}\cup\{P\}}(B) = c_{\mathcal{A}\backslash\{P\}}(B)$ for all $B \in \mathcal{K}$, hence P is a redundant frame given U and $\mathcal{A}$.

Step 2. Assume that, given a non-degenerate vNM expected utility $U(\cdot)$, frame P is redundant for a collection $\mathcal{A}$. Since $c_{\mathcal{A}\cup\{P\}}(B) = c_{\mathcal{A}\backslash\{P\}}(B) \cup c_{\{P\}}(B) = c_{\mathcal{A}\backslash\{P\}}(B)$ for any $B \in \mathcal{K}$, then for any $B \in \mathcal{K}$ and $f \in B$ we have

$$W_P(f) \geq W_P(g)\ \forall g \in B \iff f \in c_{\{P\}}(B) \implies$$

$$\implies f \in c_{\mathcal{A}\backslash\{P\}}(B) \implies \exists P' \in \mathcal{A}\backslash\{P\}: \;\; W_{P'}(f) \geq W_{P'}(g)\ \forall g \in B$$

Since $U(\cdot)$ is non-degenerate, WLOG $1 = U(x) > U(y) = 0$ for some $x, y \in \triangle X$. The above equation is true in particular for menus consisting only of canonical acts $f : S \to \triangle(\{x,y\})$. Thus we assume for now that $H = [0,1]^{|S|}$. Recall the definition of sets $Z_\lambda, O \subset R^{|S|}$ given by eqs. (8), (11) and support functional $I_P(\cdot)$ given by eq. (15). Denote

$$B \;=\; \{\mathbf{z} \in H \mid I_P(\mathbf{z}) \leq 0.5\} \quad , \quad \mathbf{z}^{\mathbf{v}} = (0.5 - 0.01 I_P(\mathbf{v}))\iota + 0.01\mathbf{v} \in Z_{0.2} \subset Z_1 = H \quad \text{for } \mathbf{v} \in O$$

where we used $|I_P(\mathbf{v})| \leq |\mathbf{v}| = 1$ implied by the properties of $I_P(\cdot) \in \mathcal{I}$ proven in lemma A.6. Recall that with the normalization $U(x) = 1, U(y) = 0$ we have $W_P(\cdot) = I_P(\cdot)$ for acts in $H = [0,1]^{|S|}$. Then for all $\mathbf{v} \in O$ we have $I_P(\mathbf{z}^{\mathbf{v}}) = 0.5$, thus $\mathbf{z}^{\mathbf{v}} \in c_{\{P\}}(B) \subseteq c_{\mathcal{A}\backslash\{P\}}(B)$. Therefore, for every $\mathbf{v} \in O$ there is $P^{\mathbf{v}} \in \mathcal{A}\backslash\{P\}$ such that $\mathbf{z}^{\mathbf{v}} \in c_{\{P^{\mathbf{v}}\}}(B)$.

Step 2.1. Assume for a contradiction that $I_{P^{\mathbf{v}}}(\mathbf{z}^{\mathbf{v}}) < 0.5$. Then $I_{P^{\mathbf{v}}}(0.5\cdot\mathbf{z}^{\mathbf{v}}+0.5\cdot 0.5\iota) > I_{P^{\mathbf{v}}}(\mathbf{z}^{\mathbf{v}})$, but $I_P(0.5\cdot\mathbf{z}^{\mathbf{v}} + 0.5\cdot 0.5\iota) = 0.5$ and $0.5\cdot\mathbf{z}^{\mathbf{v}} + 0.5\cdot 0.5\iota \in Z_1 = H$, hence $0.5\cdot\mathbf{z}^{\mathbf{v}} + 0.5\cdot 0.5\iota \in B$, therefore $\mathbf{z}^{\mathbf{v}} \notin c_{\{P^{\mathbf{v}}\}}(B)$, contradiction.

Step 2.2. Assume for a contradiction that $I_{P^{\mathbf{v}}}(\mathbf{z}^{\mathbf{v}}) > 0.5$. Then $(0.5 - 0.02 I_P(\mathbf{v})) + 0.02\mathbf{v} \in H$, and $I_P((0.5 - 0.02 I_P(\mathbf{v}))\iota + 0.02\mathbf{v}) = 0.5$, hence $(0.5 - 0.02 I_P(\mathbf{v}))\iota + 0.02\mathbf{v} \in F$. Next, since $\mathbf{z}^{v} = 0.5\cdot((0.5 - 0.02 I_P(\mathbf{v}))\iota + 0.02\mathbf{v}) + 0.5\cdot 0.5\iota$, then $I_{P^{\mathbf{v}}}((0.5 - 0.02 I_P(\mathbf{v}))\iota + 0.02\mathbf{v}) > I_{P^{\mathbf{v}}}(\mathbf{z}^{\mathbf{v}})$. We conclude that $\mathbf{z}^{\mathbf{v}} \notin c_{\{P^{\mathbf{v}}\}}(B)$, contradiction.

Step 2.3. We conclude that $I_{P^{\mathbf{v}}}(\mathbf{z}^{\mathbf{v}}) = 0.5$. Notice that for arbitrary $\mathbf{v}' \in O$ we have $I_P(\mathbf{z}^{\mathbf{v}'}) = 0.5$, hence $\mathbf{z}^{\mathbf{v}'} \in B$. Therefore, $I_{P^{\mathbf{v}}}(\mathbf{z}^{\mathbf{v}'}) \leq I_{P^{\mathbf{v}}}(\mathbf{z}^{\mathbf{v}}) = 0.5$. For arbitrary $\mathbf{v} \in O$ consider

$$\mathcal{C}^{\mathbf{v}} \;=\; \left\{ P' \in \mathcal{A}\backslash\{P\} \;\middle|\; I_{P'}(\mathbf{z}^{\mathbf{v}}) = 0.5 \text{ and } I_{P'}(\mathbf{z}^{\mathbf{v}'}) \leq 0.5 \;\forall \mathbf{v}' \in O \right\} \;\neq\; \varnothing$$

where $\mathcal{C}^{\mathbf{v}} \neq \varnothing$ for all $\mathbf{v} \in O$ by our calculations. Then for all $\mathbf{v} \in O$

$$I_P(\mathbf{z}^{\mathbf{v}}) \;=\; 0.5 \;=\; \max_{\mathbf{v}' \in O} \max_{P' \in \mathcal{C}^{\mathbf{v}'}} I_{P'}(\mathbf{z}^{\mathbf{v}}) \;=\; \max_{P' \in \mathcal{C}} I_{P'}(\mathbf{z}^{\mathbf{v}}) \quad , \quad \text{where} \;\; \mathcal{C} \;=\; \bigcup_{\mathbf{v}' \in O} \mathcal{C}^{\mathbf{v}'} \;\subseteq\; \mathcal{A}\backslash\{P\}$$

Step 2.4. Now consider arbitrary $\mathbf{z} \in R^{|S|}$ and recall its decomposition $\mathbf{z} = \overline{z}\cdot\iota + \mathbf{z}_\perp$ given by eq. (14). If $\mathbf{z}_\perp = \mathbf{0}$, then $I_P(\mathbf{z}) = \overline{z} = \max_{P' \in \mathcal{C}} I_{P'}(\mathbf{z})$. Otherwise $\frac{\mathbf{z}_\perp}{|\mathbf{z}_\perp|} \in O$, and

$$I_P(\mathbf{z}) \;=\; \overline{z} + |\mathbf{z}_\perp| \cdot I_P\left(\frac{\mathbf{z}_\perp}{|\mathbf{z}_\perp|}\right) \;=\; \overline{z} + |\mathbf{z}_\perp| \cdot \max_{P' \in \mathcal{C}} I_{P'}\left(\frac{\mathbf{z}_\perp}{|\mathbf{z}_\perp|}\right) \;=\; \max_{P' \in \mathcal{C}} I_{P'}(\mathbf{z})$$

We conclude that P is a non-trivial coherent intersection of $\mathcal{C}$.

Step 3. Let $(U, \mathcal{B})$ be the maximum framed ambiguity representation of $c(\cdot)$ existing by Theorem 2, and $(U', \mathcal{A}')$ be any other framed ambiguity representation of $c(\cdot)$. By Theorem 2, it is WLOG to consider $U' = U$. Assume $P \in \mathcal{B}\backslash\mathcal{A}'$. Then for any $B \in \mathcal{K}$

$$c_{\mathcal{A}\backslash\{P\}}(B) \;=\; c_{\mathcal{A}}(B) \;\subseteq\; c_{\mathcal{A}\cup\{P\}}(B) \;\subseteq\; c_{\mathcal{B}}(B) \;=\; c_{\mathcal{A}}(B) \;=\; c_{\mathcal{A}\backslash\{P\}}(B)$$

Therefore, by Step 2 P is a non-trivial coherent intersection of a subcollection $\mathcal{C} \subseteq \mathcal{A}$. From the other hand, if $P \in \mathcal{B} \cap \mathcal{A}'$, then P is a trivial coherent intersection of $\mathcal{C} = \{P\} \subseteq \mathcal{A}'$. We conclude that $\mathcal{B}$ consists of coherent intersections of subcollection of sets of beliefs in $\mathcal{A}'$. Proposition 3 is proven. ■

Proof of Corollary 1. If all $P = \{\mu\} \in \mathcal{A}$ are singletons, then $P_1 \cap P_2 = \emptyset$ for any two different frames. Thus by Proposition 2 a non-empty set of beliefs is a coherent intersection of $\mathcal{C}$ only if $\mathcal{C} = \{P\} = \{\{\mu\}\}$ which is a trivial coherent intersection. Thus by . Finally, if all frames are singletons, then $\{\mu_1\} \cap \{\mu_2\} = \emptyset$ for any two different frames. ■

Proof of Proposition 4

Lemma 13. *Let $c(\cdot)$ satisfies axioms 1-7. Then the following conditions are equivalent for $c(\cdot)$: (i) every representation $(U, \mathcal{A})$ of $c(\cdot)$ has a unique frame: $|\mathcal{A}| = 1$ (ii) β (iii) WARP (iv) Normality (v) Direct Ambiguity Aversion (vi) Pairwise No-C-Hedging.*

Proof. Firstly note that *WARP* and β are equivalent under axioms 1-7, since one of the axioms is α. We then are going to show the following implications: (1) $|\mathcal{A}| = 1 \implies$ *Direct Ambiguity Aversion*, *Pairwise No-C-Hedging*, *WARP* $\implies$ *Normality*, and (2) $|\mathcal{A}| > 1 \implies$ violation of *WARP*, violation of *Direct Ambiguity Aversion*, violation of *No-C-Hedging* $\implies$ violation of *Normality*. Taken together, these implications are equivalent to the statement of the Lemma.

Step 1. Let $\mathcal{A} = \{P\}$. Notice that $U(\cdot)$ is non-degenerate, otherwise $c(A) = A$ for all $A \in \mathcal{K}$, and *C-Non-Degeneracy* fails. Then

$$W_P(\lambda f + (1-\lambda)g) \geq \min\{W_P(f), W_P(g)\} \quad , \quad W_P(\lambda f + (1-\lambda)p) = \lambda W_P(f) + (1-\lambda)U(p)$$

for any $f, g \in H$, $p \in H_0$ and $\lambda \in (0,1)$. Take any $A \in \mathcal{K}$ and suppose $f \in c(A \cup \{f\})$, $g \in c(A \cup \{g\})$. Then $W_P(f), W_P(g) \geq W_P(h)$ for all $h \in A$. Therefore, $W_P(\lambda f + (1-\lambda)g) \geq W_P(\lambda f + (1-\lambda)g) \geq \min\{W_P(f), W_P(g)\} \geq W_P(h)$ for all $h \in A$. Hence, $\lambda f + (1-\lambda)g \in c(A \cup \{\lambda f + (1-\lambda)g\})$, and *Direct Ambiguity Aversion* holds. Suppose that $h \in c(\{h, f\})$ and $h \in c(\{h, p\})$. Then $W_P(h) \geq W_P(f), U(p)$. Therefore, $W_P(h) \geq \lambda W_P(f) + (1-\lambda)U(p) = W_P(\lambda f + (1-\lambda)p)$, $h \in c(\{h, \lambda f + (1-\lambda)p\})$, and *Pariwise No-C-Hedging* holds. Notice that with a single frame P choice correspondence $c(\cdot)$ is a result of the maximization of utility function W_P, hence it satisfies *WARP*. *Normality* follows from $f \in c(A) \Longleftrightarrow W_P(f) \geq W_P(g)\ \forall g \in A \Longleftrightarrow f \in c(\{f,g\}) \forall g \in A$.

Step 2. Assume $|\mathcal{A}| > 1$. Then there are $P_1, P_2 \in \mathcal{A}$ and act $f \in H$ such that $W_{P_1}(f) > W_{P_2}(f)$, otherwise $P_1 = P_2$ for all $P_1, P_2 \in \mathcal{A}$. It is easy to see that there are constant acts q_1, q_2 such that $W_{P_1}(f) = U(q_1)$ and $W_{P_2}(f) = U(q_2)$. Indeed, take $\underline{q} = f\left(\arg\min_{s\in S} U(f(s))\right)$ and $\overline{q} = f\left(\arg\max_{s\in S} U(f(s))\right)$, then $U(\underline{q}) \leq W_{P_2}(f) < W_{P_1}(f) \leq U(\overline{q})$, and we can take q_1 and q_2 as convex combinations of $\underline{q}$ and $\overline{q}$ (choice of $s \in$ argmin..., if it is not a singleton does not matter).

Step 2.1. Notice that $c(\{q_2, f\}) = \{q_2, f\}$, but $c(\{q_1, q_2, f\}) = \{q_1, f\}$, since $W_P(q_1) = U(q_1) > U(q_2) = W_P(q_2)$ for any $P \in \mathcal{A} \subseteq \Pi$. Thus, condition β is violated: $\{q_2, f\} \cap c(\{q_1, q_2, f\}) = \{f\} \neq \emptyset$, but $q_2 \in c(\{q_2, f\}) \not\subseteq c(\{q_1, q_2, f\})$. Therefore, *WARP* is violated.

Step 2.2. Now take $g = 0.25q_1 + 0.75q_2$ and consider $h' = 0.5f + 0.25q_1 + 0.25q_2$, $h = 0.5f + 0.5g = 0.5f + 0.125q_1 + 0.375q_2$. Let menu $A = \{h'\}$. Then $W_{P_1}(f) = U(q_1) \geq 0.75U(q_1) + 0.25U(q_2) = W_{P_1}(h')$, hence $f \in c(\{A \cup \{f\}\})$. Next, $W_{P_2}(h') = 0.25U(q_1) + 0.75U(q_2) = W_{P_2}(g)$, hence $g \in c(A \cup \{g\})$. However, h' strictly state-wise dominates h, therefore by *Strict Monotonicity* $h \notin c(\{h', h\}) = c(\{A \cup \{h\}\})$. Therefore, *Direct Ambiguity Aversion* is violated.

Step 2.3. Consider $p = 0.5q_1 + 0.5q_2 \in H_0$. Notice that $h' = 0.5f + 0.5p$. We have: $W_{P_2}(h) = W_{P_2}(0.5f + 0.5(0.25q_1 + 0.75q_2)) = 0.125U(q_1) + 0.875U(q_2) \geq U(q_2) = W_{P_2}(f)$, hence $h \in c(\{h, f\})$. Next, $W_{P_1}(h) = 0.625U(q_1) + 0.375U(q_2) \geq 0.5U(q_1) + 0.5U(q_2) = U(p)$, hence $h \in c(\{h, p\})$. However, $h \notin c(\{h, h'\}) = c(\{h, 0.5f + 0.5p\})$. Thus, *No-C-Hedging** is violated.

Step 2.4. Finally, suppose that *Normality* holds. Recall that *No-C-Hedging* and condition α hold by the assumption of Proposition 4. Consider arbitrary $h, f \in H$ and $p \in H_0$ and $\lambda \in (0.1)$. Suppose that $h \in c(\{f, h\})$ and $h \in c(\{h, p\})$. Then $h \succeq_c f$ and $h \succeq_c p$, hence by *Normality* we have $h \in c(\{h, f, p\})$. Therefore, by *No-C-Hedging* we get $h \in c(\{h, f, p, \lambda f + (1-\lambda)p\})$. Hence, by condition α we have $h \in c(\{h, \lambda f + (1-\lambda)p\})$. We conclude that *No-C-hedging** holds, in a contradiction to what we have proven before. ■

Proposition 4 follows from lemma 13. ■

Proof of Proposition 5

Lemma 14. *Let c_1 and c_2 have framed ambiguity representations $(U, \mathcal{A}_1)$ amd $(U, \mathcal{A}_2)$. Assume that for all $f, g \in H$ we have $c_2(\{f, g\}) = \{f\} \implies c_1(\{f, g\}) = \{f\}$. Then U_1 and U_2 are positive affine transformations of each other.*

Proof Let x maximizes U_2 and y minimizes U_2 over $p \in \triangle X$ (exist by continuity of $U(\cdot)$ and compactness of $\triangle X$), then $U_2(x) > U_2(y)$ by non-degeneracy. Assume that $U_1(\overline{x}) > U_1(x)$. Then for small enough $\epsilon > 0$ we have $c_1(\{\epsilon y + (1-\epsilon)\overline{x}, x\}) = \{\epsilon y + (1-\epsilon)\overline{x}\}$, but $c_2(\{\epsilon y + (1-\epsilon)\overline{x}, x\}) = \{x\}$, contradiction. Similarly, if $U_1(\overline{y}) < U_1(y)$, then for small enough $\epsilon > 0$ we have $c_1(\{\epsilon x + (1-\epsilon)\overline{y}, y\}) = \{y\}$, but $c_2(\{\epsilon x + (1-\epsilon)\overline{y}, y\}) = \{\epsilon x + (1-\epsilon)\overline{y}\}$, contradiction. Using continuity of the vNM expected utility function, we conclude that for any $p \in \triangle X$ there is $\lambda(p) \in [0,1]$ such that $U_2(p) = \lambda(p)U_2(x) + (1-\lambda(p))U_2(y)$, and it is easy to see that $U_1(p) = \lambda(p)U_1(x) + (1-\lambda(p))U_1(y)$ (apply $c_2(\{f, g\}) = \{f\} \implies c_1(\{f, g\}) = \{f\}$ for choices from menus $\{p, \lambda x + (1-\lambda)y\}$ with $\lambda \in [0,1]$). Normalizing $U_1(x) = U_2(x) = 1$, $U_1(y) = U_2(y) = 0$ we get $U_1(p) = U_2(p)$ for all $p \in \triangle X$. ■

If U_1 is a positive affine transformation of U_2 and $\mathcal{A}_1 \subseteq (\mathcal{A}_2)_{coh}$, then for all $B \in \mathcal{K}$ we have

$$c_1(B) \ = \ \bigcup_{P \in \mathcal{A}_1} \arg\max_{f \in B} W_P(f) \ \subseteq \ \bigcup_{P \in (\mathcal{A}_2)_{coh}} \arg\max_{f \in B} W_P(f) \ = \ c_2(B)$$

Hence, DM 1 is more decisive than DM 2. Conversely, assume that DM 1 is more decisive than DM 2. Then by Lemma 14 U_1 is a positive affine transformation of U_2. In particular, $(U_2, \mathcal{A}_1)$ represents $c_1(\cdot)$ as well, and we can assume that W_P is evaluated using vNM expected utility U_2 and set of beliefs $P \in \Pi$. Since

$$\bigcup_{P \in \mathcal{A}_1} \arg\max_{f \in B} W_P(f) \ = \ c_1(B) \ \subseteq \ c_2(B) \ = \ \bigcup_{P \in (\mathcal{A}_2)_{coh}} \arg\max_{f \in B} W_P(f)$$

then $(U_2, \mathcal{A}_1 \cup (\mathcal{A}_2)_{coh})$ represents $c_2(\cdot)$. Therefore, by Proposition 3 we have $\mathcal{A}_1 \subseteq \mathcal{A}_1 \cup (\mathcal{A}_2)_{coh} \subseteq (\mathcal{A}_2)_{coh}$. ■

Proof of Theorem 3

Throughout the proof $I_P(\cdot)$ is a support functional of set of beliefs P defined by eq. (16). We'll firstly prove the supporting lemmas, and then will prove implications $(i) \Longrightarrow (iii) \Longrightarrow (ii) \Longrightarrow (i)$ for the statements of Theorem 3.

Lemma 15. *If $c(\cdot)$ has a framed ambiguity representation with finite number of frames, then any framed ambiguity representation of $c(\cdot)$ has a finite number of frames.*

Proof. Let $(U, \mathcal{A})$ be a representation of $c(\cdot)$ with finite number of frames $|\mathcal{A}| < \infty$ and non-degenerate $U(\cdot)$. Then $|(\mathcal{A})_{coh}| \leq 2^{|\mathcal{A}|} < \infty$ from eq. (3). Then we can use Proposition 3 to conclude the proof. ∎

Recall Definition 10: DM 1 is more consistent than DM 2 if for all $A \in \mathcal{K} \quad |c_2(A)| = 1 \implies |c_1(A)| = 1$.

Lemma 16. *Let $c_1(\cdot)$ and $c_2(\cdot)$ have framed ambiguity representations. Then the following statements are equivalent:*

(i) For all $A \in \mathcal{K} \quad |c_2(A)| = 1 \implies |c_1(A)| = 1$.

(ii) For all $A \in \mathcal{K}$ and $f \in H \quad c_2(A) = \{f\} \implies c_1(A) = \{f\}$.

(iii) For all $f, g \in H \quad c_2(\{f, g\} = \{f\} \implies c_1(\{f, g\}) = \{f\}$.

Proof. Implications (ii) $\Longrightarrow$ (i), (iii) are obvious. Let $(U_1, \mathcal{A}_1)$ and $(U_2, \mathcal{A}_2)$ be framed ambiguity representations $c_1(\cdot)$ and $c_2(\cdot)$. Given a choice correspondence $c(\cdot)$, denote $\succeq_c$ to be its induced binary relation given by $f \succeq_c g$ if $f \in c(\{f, g\})$; let $\succ_c$ be the strict part of $\succeq_c$.

Step 1. (iii) $\Longrightarrow$ (ii). By Lemma 14 U_1 and U_2 are positive affine transformations of each other. Let $W_P(\cdot)$ be a utility function of the maxmin model with founded above vNM utility function $U_1 = U_2$ and set of beliefs $P \in \Pi$. If $c_2(A) = \{f\}$, then by α we have $f \succ_{c_2} g$ for all $g \in A \backslash \{f\}$. Therefore by (iii) we have $f \succ_{c_1} g$ for all $g \in A \backslash \{f\}$. It follows that $W_P(f) > W_P(g)$ for all $g \in A \backslash \{f\}$ for all $P \in \mathcal{A}_1$, hence $c_1(A) = \{f\}$, and (ii) holds.

Step 2. (i) $\Longrightarrow$ (ii). Assume that (i) holds. Let $x \in \triangle X$ be a maximizer and $y \in \triangle X$ be a minimizer of U_1 over $\triangle X$. Let $p = 0.5x + 0.5y$. Assume for a contradiction that (ii) fails for DM 1 and DM 2, then there is a menu A such that $c_2(A) = \{f\}$ and $c_1(A) = \{g\}$ with $g \neq f$. By *C-Independence* it is equivalent to $c_2(A_{0.5}) = \{f_{0.5}\}$ and $c_1(A_{0.5}) = \{g_{0.5}\}$, where $A_{0.5} =$

$0.5A + 0.5p$, $f_{0.5} = 0.5f + 0.5p$, etc. Let $W_P(\cdot)$ be the maxmin utility function with vNM expected utility $U_1(\cdot)$ and set of beliefs $P \in \Pi$, and similarly $\overline{W}_P(\cdot)$ for $U_2(\cdot)$ and $P \in \Pi$. Then $\overline{W}_P(f_{0.5}) > \overline{W}_P(g_{0.5})$ for all $P \in \mathcal{A}_2$. Since $\mathcal{A}_2$ is closed and $W_P(g_{0.5})$ is continuous in both P and $g_{0.5}$, then $\exists \epsilon > 0$ such that $\overline{W}_P(f_{0.5}) > \overline{W}_P(h)$ for all $h \in H :\ d(h, g_{0.5}) < \epsilon$. Thus, $c_2(A_{0.5} \cup \{h\}) = \{f_{0.5}\}$ for any $h \in V_\epsilon(g_{0.5}) = \{h \in H \mid d(h, g_{0.5}) < \epsilon\}$. Consider $\zeta :\ V_\epsilon(g_{0.5}) \to R$ given by $\zeta(h) = \min\limits_{P \in \mathcal{A}_1} (W_P(g_{0.5}) - W_P(h))$. Clearly, ζ is a continuous function. Next, for big enough $D > 0$ act $(1 - \epsilon/D)g_{0.5} + (\epsilon/D)x$ strictly state-wise dominates $g_{0.5}$ and belongs to $V_\epsilon(g_{0.5})$. Similarly, $(1 - \epsilon/D)g_{0.5} + (\epsilon/D)y$ is strictly state-wise dominated by $g_{0.5}$ and also belongs to $V_\epsilon(g_{0.5})$. Thus, ζ takes positive and negative values on $V_\epsilon(g_{0.5})$, hence $\zeta(h) = 0$ for some $h \in V_\epsilon(g_{0.5})$, because $V_\epsilon(g_{0.5})$ is a connected set. Since $\mathcal{A}_1$ is closed, it follows that there exists $P \in \mathcal{A}_1$ such that $W_P(h) = W_P(g_{0.5}) \geq W_P(h')$ for any $h' \in A$. Therefore, $\{h, g_{0.5}\} \in c_1(A_{0.5} \cup \{h\})$, contradicting $|c_2(A_{0.5} \cup \{h\})| = 1 \implies |c_1(A_{0.5} \cup \{h\})| = 1$. ■

Lemma 17. *Let $\mathcal{C} \subseteq \Pi$ be a non-empty closed collection of sets of beliefs. Then:*

(i) If P is a convex union of $\mathcal{C}$ given by eq. (4), then $P \in \Pi$, and $I_P(\mathbf{z}) = \min\limits_{P' \in \mathcal{C}} I_{P'}(\mathbf{z})$.

(ii) There is unique $P \in \Pi$ such that $I_P(\mathbf{z}) = \min\limits_{P' \in \mathcal{C}} I_{P'}(\mathbf{z})$. Moreover, P is a convex union of $\mathcal{C}$.

Proof of (i). Obviously, P is non-empty and convex. Next, let $\mu^n \in \bigcup\limits_{P' \in \mathcal{C}} P'$ be such that $\mu^n \longrightarrow \mu \in \Pi$. Then for every $n = 1, 2, \ldots$ there is $P^n \in \mathcal{C}$ such that $\mu^n \in P^n$. Since $\mathcal{C}$ is compact (as a closed subset of a compact set Π), there is a convergent subsequence $P^{n_k} \longrightarrow P' \in \mathcal{C}$. Thus, there is $\hat{\mu}^{n_k} \in P'$ for $k = 1, 2, \ldots$ such that $|\hat{\mu}^{n_k} - \mu^{n_k}| \longrightarrow 0$. Therefore, $|\hat{\mu}^{n_k} - \mu| \leq |\hat{\mu}^{n_k} - \mu^{n_k}| + |\mu^{n_k} - \mu| \to 0$. Since P' is closed, $\hat{\mu}^{n_k} \longrightarrow \mu \in P'$. Thus, $\bigcup\limits_{P' \in \mathcal{C}} P'$ is closed. Therefore, its convex hull P is closed (by Corollary 5.33 of Aliprantis and Border (2005)), and $P \in \Pi$. Notice that by the Theorem of Maximum the support functional $I_{P'}(\mathbf{z})$ is continuous in P', and recall that $\mathcal{C}$ is compact. Then

$$I_P(\mathbf{z}) \;=\; \min_{\mu \in P} \sum_{s \in S} \mu(s)\mathbf{z}_s \;=\; \min_{P' \in \mathcal{C}} \min_{\mu \in P'} \sum_{s \in S} \mu(s)\mathbf{z}_s \;=\; \min_{P' \in \mathcal{C}} \; I_{P'}(\mathbf{z})$$

Proof of (ii). Denote $I(\mathbf{z}) = \min\limits_{P' \in \mathcal{C}} I_{P'}(\mathbf{z})$ for $\mathbf{z} \in R^{|S|}$. Notice that $I(\cdot)$ is monotone, positively homogeneous, C-additive and normalized. Also, $I(\lambda \mathbf{z} + (1-\lambda)\mathbf{z}') = \min\limits_{P' \in \mathcal{C}} I_{P'}(\lambda \mathbf{z} + (1-\lambda)\mathbf{z}') \geq \min\limits_{P' \in \mathcal{C}} \lambda I_{P'}(\mathbf{z}) + (1-\lambda) I_{P'}(\mathbf{z}') \geq \lambda \min\limits_{P' \in \mathcal{C}} I_{P'}(\mathbf{z}) + (1-\lambda) \min\limits_{P' \in \mathcal{C}} I_{P'}(\mathbf{z}') = \lambda I(\mathbf{z}) + (1-\lambda) I(\mathbf{z}')$ for $\lambda \in (0,1)$. Thus, $I(\cdot)$ is concave. By lemma 8 there is a unique $P \in \Pi$ given by eq. (17) such that $I_P(\cdot) = I(\cdot)$. From the other hand, by (i) we have $I_{\widehat{P}}(\cdot) = I(\cdot)$, where $\widehat{P} \in \Pi$ is a convex union of $\mathcal{C}$. Therefore, by lemma 8 $P = \sigma(I) = \widehat{P}$. ■

Lemma 18. *Let $\mathcal{C} = \{P_1,...,P_N\} \subseteq \Pi$ be a non-empty finite collection of sets of beliefs, and $\lambda \in \triangle(\{1,...,N\})$. Then:*

(i) If P is a convex combination of $\mathcal{C}$ with respect to weights λ given by eq. (5), then $P \in \Pi$, and $I_P(\mathbf{z}) = \sum_{i=1}^{N} \lambda_i I_{P_i}(\mathbf{z})$

(ii) Function $\zeta : \triangle(\{1,...,N\}) \to \Pi$ given by $\zeta(\lambda) = \sum_{i=1}^{N} \lambda_i P_i$ is continuous.

(iii) There is a unique $P \in \Pi$ such that $I_P(\mathbf{z}) = \sum_{i=1}^{N} \lambda_i I_{P_i}(\mathbf{z})$. Moreover, P is a convex combination of $\mathcal{C}$ with respect to weights λ.

Proof of (i). Obviously, P is non-empty. Next, let $\mu^n \in P$ for $n = 1,2,...$, and $\mu^n \longrightarrow \mu$. Then there exists $\mu_i^n \in P_i$ for all $n = 1,2,...$, $i = 1,...,N$ such that $\mu^n = \sum_{i=1}^{N} \lambda_i \mu_i^n$ for all $n = 1,2,...$. Notice that $\times_{i=1}^{N} P_i$ is compact with respect to the product topology, thus there is a subsequence μ^{n_k} such that $\mu_i^{n_k} \longrightarrow \mu_i \in P_i$ for all $i = 1,...,N$. Then $\mu = \lim_{k\to\infty} \mu^{n_k} = \lim_{k\to\infty} \sum_{i=1}^{N} \lambda_i \mu_i^{n_k} = \sum_{i=1}^{N} \lambda_i \mu_i$, hence $\mu \in P$, and P is closed. If $\mu, \mu' \in P$, then there are $\mu_i, \mu_i' \in P_i$ for $i = 1,...,N$ such that $\mu = \sum_{i=1}^{N} \lambda_i \mu_i$ and $\mu' = \sum_{i=1}^{N} \lambda_i \mu_i'$. Since each P_i is convex, $\rho\mu_i + (1-\rho)\mu_i' \in P_i$ for all $\rho \in (0,1)$, hence $\rho\mu + (1-\rho)\mu' = \sum_{i=1}^{N} \lambda_i(\rho\mu_i + (1-\rho)\mu_i')$, and P is convex. Finally,

$$I_P(\mathbf{z}) = \min_{\mu\in P} \sum_{s\in S} \mu(s) z_s = \min_{\mu_i\in P_i \ \forall i=1,...,N} \sum_{s\in S}\sum_{i=1}^{N} \lambda_i \mu_i(s) z_s = \sum_{i=1}^{N} \lambda_i \left(\min_{\mu_i\in P_i} \sum_{s\in S} \mu_i(s) z_s \right) = \sum_{i=1}^{N} \lambda_i I_{P_i}(\mathbf{z})$$

Proof of (ii). Let $\mu \in \sum_{i=1}^{N} \lambda_i P_i$, then there is $\mu_i \in P_i$ for $i = 1,...,N$ such that $\mu = \sum_{i=1}^{N} \lambda_i \mu_i$. Consider $\mu' = \sum_{i=1}^{N} \lambda_i' \mu_i \in \sum_{i=1}^{N} \lambda_i' P_i$. Then $\mu - \mu' = \sum_{i=1}^{N} (\lambda_i - \lambda_i')\mu_i \to 0$ when $\lambda' \to \lambda$. A symmetric argument shows that for any $\mu' \in \sum_{i=1}^{N} \lambda_i' P_i$ there is $\mu \in \sum_{i=1}^{N} \lambda_i P_i$ such that $\mu' - \mu \to 0$ when $\lambda' \to \lambda$. Thus, $\sum_{i=1}^{N} \lambda_i' P_i \to \sum_{i=1}^{N} \lambda_i P_i$ with respect to the Hausdorff metric, when $\lambda' \to \lambda$. We conclude that $\zeta(\lambda) = \sum_{i=1}^{N} \lambda_i P_i$ is continuous w.r.t. $\lambda \in \triangle(\{1,...,N\})$.

Proof of (iii). Let $I(\mathbf{z}) = \sum_{i=1}^{N} \lambda_i I_{P_i}(\mathbf{z})$ for $\mathbf{z} \in R^{|S|}$. It is easy to see that $I(\cdot)$ is monotone, positively homogeneous, concave, C-additive and normalized. By lemma 8 there is a unique $P \in \Pi$ such that $I_P(\cdot) = I(\cdot)$. From the other hand, by (i) we have $I_{\widehat{P}}(\cdot) = I(\cdot)$, where $\widehat{P}$ is a convex combination of $\mathcal{C}$ with weights λ. Therefore, by lemma 8 $P = \sigma(I) = \widehat{P}$ ■

When vNM expected utility function is fixed (up to the positive affine transformation), denote $c_{\mathcal{D}}(\cdot)$ to be the choice correspondence represented by the framed ambiguity model $(U, \mathcal{D})$, and $\succeq_{\mathcal{D}}$ to be the corresponding binary relation given by $f \succeq_{\mathcal{D}} g$ if $f \in c_{\mathcal{D}}(\{f, g\})$, with $\succ_{\mathcal{D}}$ being its strict part. If $\mathcal{D} = \{P\}$ is a singleton, we'll use $\succeq_P$ instead of $\succeq_{\{P\}}$ for brevity.

Lemma 19. *Let $c_1(\cdot)$ and $c_2(\cdot)$ have framed ambiguity representations $(U_1, \mathcal{A}_1)$ and $(U_2, \mathcal{A}_2)$. Then the following statements are equivalent:*
(i) DM 1 is more consistent that DM 2.
(ii) U_1 is a positive affine transformation of U_2, and
$\left[f \succeq_{P_2} g \ \ \forall P_2 \in \mathcal{A}_2\right]$ implies $\left[f \succeq_{P_1} g \ \ \forall P_1 \in \mathcal{A}_1\right]$.
(iii) U_1 is a positive affine transformation of U_2, and
$\left[f \succ_{P_2} g \ \ \forall P_2 \in \mathcal{A}_2\right]$ implies $\left[f \succ_{P_1} g \ \ \forall P_1 \in \mathcal{A}_1\right]$.

Proof. Step 1. (i) $\Longrightarrow$ (iii). Let (i) holds. By Step 1 of the proof of lemma 16 we know that U_1 is a positive affine transformation of U_2. Next, $\left[f \succ_{P_2} g \ \ \forall P_2 \in \mathcal{A}_2\right]$ if and only if $\{f\} = c_2(\{f, g\})$, which by lemma 16 implies $\{f\} = c_1(\{f, g\})$, that is equivalent to $\left[f \succ_{P_1} g \ \ \forall P \in \mathcal{A}_1\right]$.

Step 2. (iii) $\Longrightarrow$ (i). Let (iii) holds. Normalize $U(\cdot) = U_1(\cdot) = U_2(\cdot)$ such that $1 = U(x) \geq U(p) \geq U(y) = 0$ for all $p \in \triangle X$. Since $[f \succ_P g \ \ \forall P \in \mathcal{A}_i]$ if and only if $\{f\} = c_i(\{f, g\})$, then $\{f\} = c_2(\{f, g\})$ implies $\{f\} = c_1(\{f, g\})$. Now assume $f \in c_2(\{f, g\})$. Consider $f_{0.5} = 0.5f + 0.5(0.5x + 0.5y)$, similarly $g_{0.5}$, and $f_{0.5}^n = (1 - 1/n)f_{0.5} + (1/n)x$ for n=1,2,... . Then $\{f_{0.5}^n\} = c_2(\{f_{0.5}^n, g_{0.5}\})$, hence $\{f_{0.5}^n\} = c_1(\{f_{0.5}^n, g_{0.5}\})$. By *Act Continuity* we have $\{f_{0.5}\} \in c_1(\{f_{0.5}, g_{0.5}\})$, hence $f \in c_1(\{f, g\})$. Using lemma 16, we conclude that DM 1 is more consistent than DM 2.

Step 3. (iii) $\Longrightarrow$ (ii). Let (iii) holds. Assume $\left[f \succeq_{P_2} g \ \ \forall P_2 \in \mathcal{A}_2\right]$. Using $f_{0.5}, f_{0.5}^n, g_{0.5}$ defined in Part 2, we get $\left[f_{0.5}^n \succ_{P_2} g_{0.5} \ \ \forall P_2 \in \mathcal{A}_2\right]$, hence $\left[f_{0.5}^n \succ_{P_1} g_{0.5} \ \ \forall P_1 \in \mathcal{A}_1\right]$. By continuity of $\succeq_{P_1}$ we get $\left[f_{0.5} \succeq_{P_1} g_{0.5} \ \ \forall P_1 \in \mathcal{A}_1\right]$, which implies $\left[f \succeq_{P_1} g \ \ \forall P_1 \in \mathcal{A}_1\right]$.

Step 4. (ii) $\Longrightarrow$ (iii). Let (ii) holds. Assume $\left[f \succ_{P_2} g \ \ \forall P_2 \in \mathcal{A}_2\right]$. Notice that $\zeta : \ \Pi \times H \times H$

given by $\zeta(P,f,g) = W_P(f) - W_P(g)$ is continuous on $\mathcal{A}_2 \times H \times H$. Since $\mathcal{A}_2$ is compact (as a closed subset of a compact set Π), $\mathcal{A}_2 \times H \times H$ is compact, therefore ζ is uniformly continuous on it. Then there is $\epsilon \in (0,1)$ such that $W_P(f) > W_P(g) + 2\epsilon$ for all $P \in \mathcal{A}_2$. Recall the definition of $f_{0.5}, g_{0.5}$ from Part 2, then $W_P(f_{0.5}) > W_P(g_{0.5}) + \epsilon$ for all $P \in \mathcal{A}_2$. Consider $f_{0.5}^{\epsilon} = (1-\epsilon)f_{0.5} + \epsilon y$, then $W_P(f_{0.5}^{\epsilon}) > W_P(g_{0.5})$ for all $P \in \mathcal{A}_2$, hence $\left[f_{0.5}^{\epsilon} \succeq_{P_2} g_{0.5} \ \ \forall P_2 \in \mathcal{A}_2\right]$ and $\left[f_{0.5}^{\epsilon} \succeq_{P_1} g_{0.5} \ \ \forall P_1 \in \mathcal{A}_1\right]$, and in particular $W_{P_1}(f_{0.5}) > W_{P_1}(f_{0.5}^{\epsilon}) \geq W_{P_1}(g_{0.5})$ for all $P_1 \in \mathcal{A}_1$. It follows that $\left[f_{0.5} \succ_{P_1} g_{0.5} \ \ \forall P_1 \in \mathcal{A}_1\right]$ and $\left[f \succ_{P_1} g \ \ \forall P_1 \in \mathcal{A}_1\right]$. ■

The following lemma is closely related to lemmas 1-7 in Crès et al. (2011). The difference is that $\succeq_P$ does not satisfy EUA axiom from Crès et al. (2011) with respect to $\succeq_{P_i}$, but only the Unanimity axiom. We also borrowed some of the proofs and ideas of proofs from their lemmas 1-7.

Lemma 20. *Fix a non-degenerate vNM function U. Let $\mathcal{D} = \{P_1, \dots, P_N\} \subseteq \Pi$ be a non-empty finite collection of sets of beliefs, and $\varnothing \neq P \in \Pi$. If $\forall f, g \in H$ we have $\left[f \succeq_{P_i} g \ \forall i = 1, \dots, N\right] \implies f \succeq_P g$, then there is a function $\psi: R^N \to R$ such that $I_P(\mathbf{z}) = \psi(I_{P_1}(\mathbf{z}), \dots, I_{P_N}(\mathbf{z}))$ for all $\mathbf{z} \in R^{|S|}$. Moreover, we can choose ψ to be simultaneously:*

(i) Positively homogeneous: $\psi(\lambda t) = \lambda \psi(t)$ for all $\lambda \geq 0$.

(ii) Monotone: $t \geq t' \implies \psi(t) \geq \psi(t')$.

(iii) C-additive: $\psi(t + \beta \cdot (1, \dots, 1)) = \psi(t) + \psi(\beta \cdot (1, \dots, 1))$ for all $\beta \in R$.

(iv) Normalized: $\psi(1, \dots, 1) = 1$.

Proof. Denote $e = (1, \dots, 1) \in R^N$. Normalize vNM expected utility function $1 \geq U(x) \geq U(p) \geq U(y) = -1$ for all $p \in \triangle X$ and consider acts in the utility space $(U(f(s)))_{s \in S} = \mathbf{z} \in H = [-1,1]^{|S|}$. This way $W_{\widetilde{P}}(\mathbf{z}) = I_{\widetilde{P}}(\mathbf{z})$ for $\mathbf{z} \in [-1,1]^{|S|}$, where $I_{\widetilde{P}}(\cdot)$ is a support functional of set of beliefs $\widetilde{P}$ given by eq. (16). Define $\mathbf{I}^N = (I_1, \dots, I_N)$ for brevity. Denote also $\sim_{\widetilde{P}}$ to be the symmetric part of $\succeq_{\widetilde{P}}$.

Step 1. Take arbitrary $\mathbf{z}', \mathbf{z} \in H$. We argue that if $\mathbf{I}^N(\mathbf{z}') \geq \mathbf{I}^N(\mathbf{z})$, then $I_P(\mathbf{z}') \geq I_P(\mathbf{z})$, and if $\mathbf{I}^N(\mathbf{z}') = \mathbf{I}^N(\mathbf{z})$, then $I_P(\mathbf{z}') = I_P(\mathbf{z})$. Indeed, if $\mathbf{I}^N(\mathbf{z}') \geq \mathbf{I}^N(\mathbf{z})$, then $\mathbf{z}' \succeq_{P_i} \mathbf{z}$ for all $i = 1, \dots, N$, hence $\mathbf{z}' \succeq_P \mathbf{z}$, and $I_P(\mathbf{z}') \geq I_P(\mathbf{z})$.

Step 2. Let $Q = \mathbf{I}^N([-1,1]^{|S|})$ be the image of the set of acts $H = [-1,1]^{|S|}$ with respect to the vector of utility functions $\mathbf{I}^N = (I_{P_i})_{i=1,\dots,N}$, representing $\succeq_{P_i}$, $i = 1, \dots, N$, and let $cone(Q)$ be a minimum cone that contains Q, which is given by $cone(Q) = \{t \in R^N \mid \exists \widetilde{t} \in$

Q and $\gamma \geq 0: \ t = \gamma \tilde{t}\}$. Define function $\phi: \ cone(Q) \rightarrow R$ by

$$\phi(t) \ = \ \gamma I_P(\mathbf{z}) \ \text{ for some } \gamma > 0, \mathbf{z} \in H \text{ such that } \ \mathbf{I}^N(\mathbf{z}) = \gamma^{-1} t \qquad (22)$$

Notice that ϕ is well-defined. Indeed, firstly, let $t \in cone(Q)$, then there is $\gamma > 0$ and $\hat{t} \in Q$ such that $t = \gamma \hat{t}$ and there is $\mathbf{z} \in H$ such that $\mathbf{I}^N(\mathbf{z}) = \hat{t} = \gamma^{-1} t$. Secondly, let $\mathbf{z}, \mathbf{z}' \in H$ be such that $\mathbf{I}^N(\mathbf{z}) = \gamma^{-1} t$ and $\mathbf{I}^N(\mathbf{z}') = (\gamma')^{-1} t$ for some $\gamma, \gamma' > 0$. WLOG, $\gamma \geq \gamma'$. Then $(\gamma'/\gamma)\mathbf{z}' \in H$, $\mathbf{I}^N((\gamma'/\gamma)\mathbf{z}') = (\gamma'/\gamma)\mathbf{I}^N(\mathbf{z}') = \gamma^{-1} t = \mathbf{I}^N(\mathbf{z})$. Therefore, by Step 1 we have $(\gamma'/\gamma) I_P(\mathbf{z}') = I_P((\gamma'/\gamma)\mathbf{z}') = I_P(\mathbf{z})$. Thus, $\phi(t) = \gamma' I_P(\mathbf{z}') = \gamma I_P(\mathbf{z})$ does not depend on the choice of γ and $\mathbf{z}$ such that $\mathbf{I}^N(\mathbf{z}) = \gamma^{-1} t$.

Step 3. Notice that, since $\mathbf{I}^N(0) = 0$, and $I_P(0) = 0$, then $\phi(0) = 0$. Let us show that ϕ is positively homogeneous, monotone, C-additive and normalized. We consider arbitrary $t, t' \in cone(Q)$. Then there are $\mathbf{z}', \mathbf{z} \in H$ and $\gamma, \gamma' > 0$ such that $\mathbf{I}^N(\mathbf{z}) = \gamma^{-1} t$, $\mathbf{I}^N(\mathbf{z}') = (\gamma')^{-1} t'$, and we have $\phi(t) = \gamma I_P(\mathbf{z})$, $\phi(t') = \gamma' I_P(\mathbf{z}')$.

Step 3.1. Let $t' = \beta t$ for $\beta \geq 0$. If $t = 0$, then $t' = 0$ and $\phi(t') = 0 = \beta \cdot 0 = \beta \cdot \phi(t)$. If $\beta = 0$, then $t' = 0$, and $\phi(t') = 0 = 0 \cdot \phi(t)$. In other cases, $\beta > 0$ and $t, t' \neq 0$. Suppose that $(\gamma\beta)/\gamma' \leq 1$. Then $(\gamma\beta)/\gamma' \cdot \mathbf{z} \in H$. Next, $\mathbf{I}^N((\gamma\beta)/\gamma' \cdot \mathbf{z}) = ((\gamma\beta)/\gamma') \cdot \gamma^{-1} t = (\gamma')^{-1}\beta \cdot t = (\gamma')^{-1} t' = \mathbf{I}^N(\mathbf{z}')$. Thus, by Step 1 $(\gamma\beta)/\gamma' \cdot I_P(\mathbf{z}) = I_P((\gamma\beta)/\gamma' \cdot \mathbf{z}) = I_P(\mathbf{z}')$. Finally, $\phi(t') = \gamma' I_P(\mathbf{z}') = \gamma' \cdot (\gamma\beta)/\gamma' \cdot I_P(\mathbf{z}) = \beta\gamma I_P(\mathbf{z}) = \beta\phi(t)$. The case when $(\gamma\beta)/\gamma' > 1$ is proven by a symmetric argument, considering $(\gamma'/(\gamma\beta)) \cdot \mathbf{z}' \in H$. Thus, ϕ is positively homogeneous.

Step 3.2. Let $t \geq t'$. Assume $\gamma/\gamma' \leq 1$. Then $(\gamma/\gamma') \cdot \mathbf{z} \in H$, and $\mathbf{I}^N((\gamma/\gamma') \cdot \mathbf{z}) = (\gamma/\gamma') \cdot \mathbf{I}^N(\mathbf{z}) = (\gamma')^{-1} t \geq (\gamma')^{-1} t' = \mathbf{I}^N(\mathbf{z}')$. Thus by Step 1 $\phi(t) = \gamma I_P(\mathbf{z}) = \gamma' I_P((\gamma/\gamma') \cdot \mathbf{z}) \geq \gamma' I_P(\mathbf{z}') = \phi(t')$. We conclude that ϕ is monotone.

Step 3.3. Consider $\iota = (1, \dots, 1) \in H$. Then $\mathbf{I}^N(\iota) = e$, hence $\phi(e) = I_P(\iota) = 1$. Therefore, ϕ is normalized.

Step 3.4. Let $t = t' + \beta \cdot e$ for $\beta \in R$. If $\beta = 0$, then $\phi(t') = \phi(t) = \phi(t) + \phi(0 \cdot e)$, thus C-additivity holds. WLOG, $\beta > 0$ (if $\beta < 0$, then consider $t' = t + (-\beta) \cdot e$). Denote $\delta = \min\{\gamma'/(3\beta), 1/3\}$. Then $\delta \cdot \mathbf{z}' + \delta\beta(\gamma')^{-1} \cdot \iota \in H$. We have $\mathbf{I}^N(\delta \cdot \mathbf{z}' + \delta\beta(\gamma')^{-1} \cdot \iota) = \delta \cdot \mathbf{I}^N(\mathbf{z}') + \delta\beta(\gamma')^{-1} \cdot e = \delta(\gamma')^{-1} \cdot (t' + \beta \cdot e) = \delta(\gamma')^{-1} \cdot t$. Therefore, $\phi(t) = (\gamma'/\delta) \cdot I_P(\delta\mathbf{z}' + \delta\beta(\gamma')^{-1} \cdot \iota) =$

$(\gamma'/\delta)\cdot(\delta I_P(\mathbf{z}')+\delta\beta(\gamma')^{-1}) = \gamma' I_P(\mathbf{z}')+\beta = \phi(t')+\beta = \phi(t')+\beta\phi(e) = \phi(t')+\phi(\beta\cdot e)$ by Steps 3.3. and 3.2. Thus, ϕ is C-additive.

Step 4. Take $\gamma = 1$, then by the definition of ϕ we have $I_P(\mathbf{z}) = \phi(\mathbf{I}^N(\mathbf{z}))$ for all $\mathbf{z} \in R^{|S|}$ whenever $t = \mathbf{I}^N(\mathbf{z}) \in cone(Q)$. To see that this always holds, $\mathbf{I}^N(\mathbf{z}) = |\mathbf{z}| \cdot \mathbf{I}^N(\mathbf{z}/(|\mathbf{z}|)) \in cone(Q)$ for $|\mathbf{z}| \neq 0$, and $\mathbf{I}^N(0) \in Q \subset cone(Q)$. Thus, $I_P(\mathbf{z}) = \phi(\mathbf{I}^N(\mathbf{z}))$ for all $\mathbf{z} \in R^{|S|}$.

Step 5. Define $\psi: \ R^N \to R$ by

$$\psi(t) \quad = \quad \inf \ \{\phi(t') \mid t' \in cone(Q) \text{ and } t' \geq t\} \tag{23}$$

Notice that, since $\iota, (-\iota) \in H$, then $e, (-e) \in Q$, and $\beta \cdot e \in cone(Q)$ for all $\beta \in R$. Since $\left(\min\limits_{i=1,\dots,N} t_i\right) \cdot \ e \leq t \leq \left(\max\limits_{i=1,\dots,N} t_i\right) \cdot \ e$, then inf in eq. (23) exists, and $\psi(\cdot)$ is well-defined.

Step 5.1. Let us show that $I_P(\mathbf{z}) = \psi(\mathbf{I}^N(\mathbf{z}))$ for all $\mathbf{z} \in R^{|S|}$. We argue that $\psi(\mathbf{I}^N(\mathbf{z})) = \phi(\mathbf{I}^N(\mathbf{z}))$. Indeed, $\mathbf{I}^N(\mathbf{z}) = |\mathbf{z}| \cdot \mathbf{I}^N(\mathbf{z}/(|\mathbf{z}|)) \in cone(Q)$ for $|\mathbf{z}| \neq 0$, and $\mathbf{I}^N(0) \in Q \subset cone(Q)$, thus $\phi(\mathbf{I}^N(\mathbf{z}))$ is well-defined; if $t' \geq \mathbf{I}^N(\mathbf{z})$, then $\phi(t') \geq \phi(\mathbf{I}^N(\mathbf{z}))$, since $\phi(\cdot)$ is monotone by Step 3.2. Step 4 finishes the proof.

Step 5.2. Let $\gamma > 0$ and $t \in R^{|S|}$. Then

$$\psi(\gamma t) \ = \ \inf\{\phi(t') \mid t' \in cone(Q) \text{ and } t' \geq \gamma t\} \ = \ \inf\Big\{\phi(t') \mid t' \in cone(Q) \text{ and } \gamma^{-1}t' \geq t\Big\} \ =$$

$$= \ \inf\Big\{\phi(t') \mid \gamma^{-1}t' \in cone(Q) \text{ and } \gamma^{-1}t' \geq t\Big\} \ = \ \inf\{\phi(\gamma t'') \mid t'' \in cone(Q) \text{ and } t'' \geq t\} \ =$$

$$= \ \inf\{\gamma\phi(t'') \mid t'' \in cone(Q) \text{ and } t'' \geq t\} \ = \ \gamma \cdot \inf\{\phi(t'') \mid t'' \in cone(Q) \text{ and } t'' \geq t\} \ = \ \gamma\psi(t)$$

If $\gamma = 0$, then $\psi(\gamma t) = \psi(0) = \gamma\psi(t)$. Thus, $\psi(\cdot)$ is positively homogeneous.

Step 5.3. Let $t \geq t'$ for some $t, t' \in R^{|S|}$. Then

$$\{\ t'' \in cone(Q) \mid t'' \geq t\ \} \quad \subseteq \quad \{\ t'' \in cone(Q) \mid t'' \geq t'\ \}$$

Therefore, $\psi(t) \geq \psi(t')$. We conclude that $\psi(\cdot)$ is monotone.

Step 5.4. Since $e \in cone(Q)$, and $\phi(\cdot)$ is monotone and normalized, then $\psi(e) = \phi(e) = 1$. Hence, $\psi(\cdot)$ is normalized.

Step 5.5. Let's firstly show that $t \in cone(Q)$ implies $t + \beta e \in cone(Q)$ for all $\beta \in R$. We already know this for $t = 0$. Case $\beta = 0$ is also trivial, so consider $t \neq 0$ and $\beta \neq 0$. By the definition of $cone(Q)$ there is $\mathbf{z} \in H$ and $\gamma > 0$ such that $t = \gamma \mathbf{I}^N(\mathbf{z})$ (since $t \neq 0$, $\gamma \neq 0$). Take $\delta = \min\{\gamma/(3|\beta|), 1/3\} > 0$. Then $\delta \mathbf{z} + \delta\gamma^{-1}\beta \cdot \iota \in H$. Therefore, $\delta \mathbf{I}^N(\mathbf{z}) + \delta\gamma^{-1}\beta \cdot e = \mathbf{I}^N(\delta\mathbf{z} + \delta\gamma^{-1}\beta \cdot \iota) \in Q$. Hence, $t + \beta e = \gamma \mathbf{I}^N(\mathbf{z}) + \beta e \in cone(Q)$. To see that $\psi(\cdot)$ is C-additive,

$$\psi(t+\beta e) \ = \ \inf\{\phi(t') \mid t' \in cone(Q) \text{ and } t' \geq t + \beta e\} \ = \ \inf\{\phi(t') \mid t' \in cone(Q) \text{ and } t' - \beta e \geq t\} \ =$$

$$= \ \inf\{\phi(t') \mid t' - \beta e \in cone(Q) \text{ and } t' - \beta e \geq t\} \ = \ \inf\{\phi(t'' + \beta e) \mid t'' \in cone(Q) \text{ and } t'' \geq t\} \ =$$

$$= \ \inf\{\phi(t'') + \beta \mid t' - \beta e \in cone(Q) \text{ and } t' - \beta e \geq t\} \ = \ \inf\{\phi(t'') \mid t'' \in cone(Q) \text{ and } t'' \geq t\} + \beta \ =$$

$$= \ \psi(t) + \beta \ = \ \psi(t) + \beta\psi(e) \ = \ \psi(t) + \psi(\beta e)$$

To summarise, function $\psi: R^N \to R$ given by eq. (23) is positively homogeneous, monotone, C-additive, normalized, and satisfies $I_P(\mathbf{z}) = \psi(\mathbf{I}^N(\mathbf{z}))$ for all $\mathbf{z} \in R^{|S|}$. ■

We next want to make use results from Frick et al. (2019).

Lemma 21. *Let function $\psi: [0,1]^N \to R$ be positively homogeneous, monotone, C-additive and normalized, then there is a non-empty compact (w.r.t. Hausdorff metric) collection Θ of non-empty compact and convex sets of weights $\Lambda \subseteq \triangle(\{1,...,N\})$ such that for all $t \in [0,1]^N$*

$$\psi(t) \ = \ \max_{\Lambda \in \Theta} \min_{\lambda \in \Lambda} \sum_{i=1}^{N} \lambda_i t_i \tag{24}$$

Proof. This formula is eq. (2) in Frick et al. (2019), and later is discussed in their Remark (iv). Frick et al. (2019) give the formal proof in their Appendix B1. ■

The next lemma is Proposition 1[20] in Crès et al. (2011) formulated in the utility space:

Lemma 22. *Let $\Lambda \subseteq \triangle(1,...,N)$ be a non-empty closed and convex set of weights, then*

$$I_P(\mathbf{z}) \ = \ \min_{\lambda \in \Lambda} \sum_{i=1}^{N} \lambda_i I_{P_i}(\mathbf{z}) \ \ \forall \mathbf{z} \in R^{|S|} \quad \textit{iff} \quad P \ = \ \left\{ \mu \in \triangle S \ \middle| \ \exists \lambda \in \Lambda \textit{ and } \mu_i \in P_i : \ \mu = \sum_{i=1}^{N} \lambda_i \mu_i \right\}$$

Proof. Normalize vNM expected utility function $U(\cdot)$ such that $1 = U(x) \geq U(p) \geq U(y) = -1$ for all $p \in \triangle X$. Then use Proposition 1 from Crès et al. (2011) to argue that the statement is true if we change the domain of $I_P(\mathbf{z})$ and $I_{P_i}(\mathbf{z})$ from $R^{|S|}$ to $[-1,1]^{|S|}$. Then use

[20] We demand Λ to be non-empty, so that J_0 in eq. (5) of Crès et al. (2011) is well-defined.

positive homogeneity of I_P, I_{P_i} to argue that the statement remains true if we expand the domain from $[-1,1]^{|S|}$ to $R^{|S|}$. ■

Fix a vNM expected utility function $U(\cdot)$. For an arbitrary non-empty collection of frames $\mathcal{D}$ denote $(\mathcal{D})_{agg}$ to be its closure with respect to the operations of convex union, coherent intersection and convex combination, given by Definition 11.

Lemma 23. *For any non-empty collection of sets of beliefs $\mathcal{D} \subseteq \Pi$ its closure with respect to operations of convex union, coherent intersection and convex combination exists and is unique.*

Proof. Let $\mathbb{F}$ be the set of collections of sets of beliefs $\mathcal{F}$ such that $\mathcal{D} \subseteq \mathcal{F} \subseteq \Pi$ and $\mathcal{F}$ is closed with respect to the operations of convex union, coherent intersection and convex combination for all $\mathcal{F} \in \mathbb{F}$. Notice that $\Pi \in \mathbb{F} \neq \emptyset$. Then the unique $(\mathcal{D})_{agg}$ is given by $(\mathcal{D})_{agg} = \bigcap_{\mathcal{F} \in \mathbb{F}} \mathcal{F}$ ■

Lemma 24. *Let vNM expected utility function $U(\cdot)$ be fixed, and $\emptyset \neq \mathcal{D} \subseteq \Pi$. If $P \in (\mathcal{D})_{agg}$, then $[f \succ_{P'} g \quad \forall P' \in \mathcal{D}]$ implies $f \succ_P g$.*

Proof. Let $\mathcal{E} \subseteq (\mathcal{D})_{agg}$ be the collection of all sets of beliefs $P \in (\mathcal{D})_{agg}$ such that $[f \succ_{P'} g \quad \forall P' \in \mathcal{D}]$ implies $f \succ_P g$. Notice that $\emptyset \neq \mathcal{D} \subseteq \mathcal{E}$. Take an arbitrary non-empty closed subcollection $\mathcal{C} \subseteq \mathcal{E}$. Normalize $1 = U(x) \leq U(p) \leq U(y) = 0$ for all $p \in \triangle X$.

Case 1. Assume that P is a convex union of $\mathcal{C}$ given by eq. (4). Using lemma 17, we get $W_P(f) = \min_{P' \in \mathcal{C}} W_{P'}(f)$ for all $f \in H$.

Case 2. Assume that P is a coherent intersection of $\mathcal{C}$ given by eq. (20). Then $W_P(f) = \max_{P' \in \mathcal{C}} W_{P'}(f)$ for all $f \in H$.

Case 3. Assume P is a convex combination of $\mathcal{C} = \{P_1, ..., P_N\}$ with weights λ, given by eq. (5). Using lemma 18, we get $W_P(f) = \sum_{i=1}^{N} \lambda_i W_{P_i}(f)$ for all $f \in H$.

Now let $[f \succ_{P'} g \quad \forall P' \in \mathcal{D}]$. Since $\mathcal{C} \subseteq \mathcal{E}$, then $f \succ_{P'} g$ for all $P' \in \mathcal{C}$, thus $W_{P'}(f) > W_{P'}(g)$ for all $P' \in \mathcal{C}$, which in all three cases implies $W_P(f) > W_P(g)$ and $f \succ_P g$. Notice that since $\mathcal{C} \subseteq \mathcal{E} \subseteq (\mathcal{D})_{agg}$, and $(\mathcal{D})_{agg}$ is closed with respect to all three operations, then $P \in (\mathcal{D})_{agg}$. Therefore, $P \in \mathcal{E}$. Hence, $\mathcal{E}$ is closed with respect to all three operations. Since $\mathcal{D} \subseteq \mathcal{E} \subseteq (\mathcal{D})_{agg}$, we conclude that $\mathcal{E} = (\mathcal{D})_{agg}$, which proves the lemma. ■

Lemma 25. *Let $c_1(\cdot)$ and $c_2(\cdot)$ have framed ambiguity representations $(U_1, \mathcal{A}_1)$ and $(U_2, \mathcal{A}_2)$ with $|\mathcal{A}_2| < \infty$. Then the following statements are equivalent:*

(i) DM 1 is more consistent than DM 2.

(ii) U_1 is a positive affine transformation of U_2, and $\mathcal{A}_1$ belongs to the closure of $\mathcal{A}_2$ with respect to operations of convex union, coherent intersection and convex combination.

(iii) U_1 is a positive affine transformation of U_2, and any $P \in \mathcal{A}_1$ is a coherent intersection of convex unions of convex combinations of sets of beliefs in $\mathcal{A}_2$

Proof. Step 1. (i) $\Longrightarrow$ (iii). Suppose DM 1 is more consistent than DM 2. Let $P \in \mathcal{A}_1$. By lemma 19 U_1 is a positive affine transformation of U_2, thus we normalize $U_1(\cdot) = U_2(\cdot) = U(\cdot)$ without loss of generality. Lemma 19 also implies that for any $f, g \in H$ if $\left[f \succeq_{P_i} g \; \forall i = 1, ..., N\right]$, then $f \succeq_P g$. Therefore, by lemma 20 there is a positively homogeneous, monotone, C-additive and normalized function $\psi : R^N \to R$ such that $I_P(\mathbf{z}) = \psi(I_{P_1}(\mathbf{z}), ..., I_{P_N}(\mathbf{z}))$ for all $\mathbf{z} \in R^{|S|}$. By lemma 21 there is a non-empty compact collection $\Theta \subseteq 2^{\triangle(\{1,...,N\})}$ of non-empty compact convex sets of weights $\Lambda \subseteq \triangle(\{1, ..., N\})$ such that

$$I_P(\mathbf{z}) \;=\; \max_{\Lambda \in \Theta} \min_{\lambda \in \Lambda} \sum_{i=1}^{N} \lambda_i I_{P_i}(\mathbf{z}) \quad \forall \mathbf{z} \in R^{|S|} \tag{25}$$

Notice that by lemma 18 for arbitrary $\lambda \in \Lambda \in \Theta$ there is a unique $P^\lambda \in \Pi$ given by $P^\lambda = \sum_{i=1}^{N} \lambda_i P_i$ (eq. 5) such that $I_{P^\lambda}(\mathbf{z}) = \sum_{i=1}^{N} \lambda_i I_{P_i}(\mathbf{z})$ for all $\mathbf{z} \in R^{|S|}$. Thus, P^λ is a convex combination of P_i, $i = 1, ..., N$ with weights λ. Therefore, for arbitrary $\Lambda \in \Theta$ we have $\min_{\lambda \in \Lambda} \sum_{i=1}^{N} \lambda_i I_{P_i}(\mathbf{z}) = \min_{\lambda \in \Lambda} I_{P^\lambda}(\mathbf{z}) \; \forall \mathbf{z} \in R^{|S|}$. For arbitrary $\Lambda \in \Theta$ consider $\mathbb{D}_\Lambda \equiv \left\{ \sum_{i=1}^{N} \lambda_i P_i \right\}_{\lambda \in \Lambda} = \left\{P^\lambda\right\}_{\lambda \in \Lambda} \subseteq \Pi$. Since Λ is compact, and since by lemma 18 $\sum_{i=1}^{N} \lambda_i P_i$ is continuous with respect to $\lambda \in \triangle(\{1, ..., N\})$, then $\mathbb{D}_\Lambda$ is compact. Therefore, by lemma 17 there is a unique P^Λ given by a convex union of P^λ, $\lambda \in \Lambda$ (eq. 4) such that

$$I_{P^\Lambda}(\mathbf{z}) \;=\; \min_{\lambda \in \Lambda} I_{P^\lambda}(\mathbf{z}) \;=\; \min_{\lambda \in \Lambda} \sum_{i=1}^{N} \lambda_i I_{P_i}(\mathbf{z}) \quad \forall \mathbf{z} \in R^{|S|} \tag{26}$$

Therefore,

$$I_P(\mathbf{z}) \;=\; \max_{\Lambda \in \Theta} I_{P^\Lambda}(\mathbf{z}) \quad \forall \mathbf{z} \in R^{|S|} \tag{27}$$

We conclude that any $P \in \mathcal{A}_1$ is a coherent intersection of sets of beliefs P^Λ, $\Lambda \in \Theta$, where each P^Λ is a convex union of sets of beliefs P^λ, $\lambda \in \Lambda$, and, in turn, each P^λ is a convex combination of $P_i \in \{P_1, ..., P_N\} = \mathcal{A}_2$ with weights λ.

Proof. Step 2. (iii) $\Longrightarrow$ (ii). The statement is obvious.

Step 3. (ii) $\Longrightarrow$ (i). By lemma 24 for any $P \in \mathcal{A}_1$ for any $f, g \in H$ if $[f \succ_{P'} g \; \forall P' \in \mathcal{A}_2]$, then $f \succ_P g$. Therefore, by lemma 19 DM 1 is more consistent than DM 2. ∎

Theorem 3 follows from lemma 25. ∎

Proof of Proposition 6, Corollary 2 and Corollary 3

Proposition 6 follows from lemma 16. ∎

Proof of Corollary 2, Part 1. Assume firstly that $c(\{f, g\}) = c'(\{f, g\})$, then by Proposition 6 DM with choice correspondence $c(\cdot)$ is more consistent than DM' with choice correspondence $c'(\cdot)$, and vice versa DM' is more consistent than DM. Therefore, by Theorem 3 $U(\cdot)$ and $U'(\cdot)$ are positive affine transformations of each other, and $\mathcal{A} \subseteq (\mathcal{A}')_{agg}$ and $\mathcal{A}' \subseteq (\mathcal{A})_{agg}$, where $(\cdot)_{agg}$ denotes the closure with respect to the operations of convex union, convex combination and coherent intersection. Observe that $(\cdot)_{agg}$ is monotone with respect to set inclusion, thus $(\mathcal{A})_{agg} \subseteq ((\mathcal{A}')_{agg})_{agg} = (\mathcal{A}')_{agg}$ and $(\mathcal{A}')_{agg} \subseteq ((\mathcal{A})_{agg})_{agg} = (\mathcal{A})_{agg}$. Therefore, $(\mathcal{A})_{agg} = (\mathcal{A}')_{agg}$.

Part 2. Assume now that U and U' are positive affine transformations of each other, and $(\mathcal{A})_{agg} = (\mathcal{A}')_{agg}$. Then $\mathcal{A} \subseteq (\mathcal{A}')_{agg}$ and $\mathcal{A}' \subseteq (\mathcal{A})_{agg}$, and DM is more consistent than DM', and DM' is more consistent than DM by Theorem 3. Therefore, by Proposition 6 for all $f, g \in H$ we have $c(\{f, g\}) \subseteq c'(\{f, g\})$ and $c'(\{f, g\}) \subseteq c(\{f, g\})$. We conclude that $c(\{f, g\}) = c'(\{f, g\})$ for all binary menus $A = \{f, g\}$. ∎

Proof of Corollary 3. Consider DM 2 with a with choice correspondence $c_2(\cdot)$ represented by a framed ambiguity model $(U, \{P_i\}_{i=1,\dots,N})$, where $U(\cdot)$ is a common vNM expected utility function that is a part of representation of $\succeq_i$, $i = 1, \dots, N$ and $\succeq$, and DM 1 with with choice correspondence $c_1(\cdot)$, represented by $(U, \{P\})$. These models are indeed well-defined, since $U(\cdot)$ is non-degenerate because of non-degeneracy of $\succeq_i$, $\succeq$, and families of frames are finite. Therefore, by lemma 19 $\succeq$ satisfies *Unanimity* with respect to $(U, \{P_i\}_{i=1,\dots,N})$ if and only if DM 1 is more consistent than DM 2, which is equivalent by

Theorem 3 to $\{P\} \subseteq (\{P_1, ..., P_N\})_{agg}$ ■

Proof of Proposition 7

Lemma 26. *Let Condition 1 or 2 hold. Then*

$$I_{\bigcap\limits_{i=1,...,N} P_i}(\mathbf{z}) \quad = \quad sup\left\{\sum_{i=1}^{N} I_{P_i}(\mathbf{z}_i) \;\middle|\; \sum_{i=1}^{N} \mathbf{z}_i = \mathbf{z}\right\} \tag{28}$$

where for each $\mathbf{z} \in R^{|S|}$ the suprenum is attained, and $I_P(\cdot)$ is a support functional of set P given by eq. 16[21].

Proof. If Condition 2 holds, the statement follows from Corollary 16.4.1 in Rockafellar (1970). If Condition 1 holds, the statement follows from Theorem 20.1 in Rockafellar (1970) for the indicator functions $f_i(\cdot) = \delta(\cdot|P_i)$, where $f_i(\cdot)$ are polyhedral because P_i are polyhedral (Corollary 19.2.1 in Rockafellar (1970)). ■

Now let's prove Proposition 7. Rename $\mathcal{A}_1 = \{P_1\}$ to $\mathcal{A}_1 = \{P\}$ for convenience and denote $\mathcal{A}_2 = \{P_1, ..., P_N\}$.

Part 1. Step 1. Assume that U_1 is a positive affine transformation of U_2, and $P = \bigcap\limits_{i=1}^{N} P_i$. WLOG, $U_1(\cdot) = U_2(\cdot) = U(\cdot)$ normalized such that $U(x) = 1$ and $U(y) = 0$, where x is the best, and y is the worst prize, and we use this vNM utility function in our calculations. Take $q = 0.5x + 0.5y$.

Step 2. Consider arbitrary $f \in H$. By lemma 26 there exist $z_1, ..., z_N \in R^{|S|}$ such that $W_P(U(f)) = \sum\limits_{i=1}^{N} I_{P_i}(z_i)$ and $U(f) = \sum\limits_{i=1}^{N} (z_i)$, where $U(f)_s = U(f(s))$.

Consider $\tilde{z}_i = z_i + \left(\sum\limits_{j=1}^{N} I_{P_j}(z_j) - I_i(z_i)\right)\iota$. Since $I_{P_i}(\cdot)$ is C-additive and normalized, then $z = \sum\limits_{i=1}^{N} \tilde{z}_i$, $W_P(U(f)) = \sum\limits_{i=1}^{N} I_{P_i}(\tilde{z}_i)$, and $I_{P_i}(\tilde{z}_i) = W_P(U(f))$ for all $i = 1, ..., N$.

Take $\lambda = 0.1\left(\max\left\{1, \max\limits_{i=1,...,N} |\tilde{z}_i|\right\}\right)^{-1} > 0$. Then $\hat{z}_i \equiv \lambda\tilde{z}_i + (1-\lambda)0.5\iota \in Z_{0.1}$ and $\lambda U(f) + (1-$

[21] Notice that we define support functional as a minimum of a linear function over a set, while Rockafellar (1970) defines it as a maximum. Hence we get sup instead of his inf in lemma 28.

$\lambda)0.5\iota = \sum_{i=1}^{N} \hat{z}_i$, and by positive homogeneity and C-additivity of the support functional we have $I_P(\lambda U(f) + (1-\lambda)0.5\iota) = \sum_{i=1}^{N} I_{P_i}(\hat{z}_i)$. Consider $f_i \in H$, $i = 1,...,N$ given by

$$f_i(s) \;=\; \frac{\lambda}{N} f(s) + \Big((\hat{z}_i)_s - \frac{\lambda}{N} U(f(s))\Big)x + \Big(1 - (\hat{z}_i(s))_s + \frac{\lambda}{N} U(f(s))\Big)y$$

Notice that because of our choice of λ we have $(\hat{z}_i)_s - \frac{\lambda}{N} U(f(s)) \in [0.3, 0.7]$, thus f_i are valid acts. It is easy to see that $\lambda f + (1-\lambda)q = \sum_{i=1}^{k} \sigma_i f_i$ for $\sigma_i = 1/N$. Next, $U(f_i(s)) = U((\hat{z}_i)_s)$, thus $W_{P_i}(f_i) = I_{P_i}(U(f_i)) = I_P(\lambda U(f) + (1-\lambda)0.5\iota) = W_P(\lambda f + (1-\lambda)q)$ for all $i = 1,...,N$. Since $\lambda f + (1-\lambda)q \in c_1(\{\lambda f + (1-\lambda)q, p\}) \Longleftrightarrow W_P(\lambda f + (1-\lambda)q) \geq U(p)$, then $W_{P_i}(f_i) \geq U(p)$, thus $f_i \in c_2(\{f_i, p\})$.

Step 3. Take σ_i, f_i and λ from the previous step. Assume $f_i \in c_2(\{f_i, p\})$ for all $i = 1,...,k$. Then $W_{\bigcap_{i=1}^{N} P_i}(f_j) \geq \max_{i=1,...,N} W_{P_i}(f_j) \geq U(p)$. Therefore, by quasi-concavity of $W(\cdot)$ we have $W_{\bigcap_{i=1}^{N} P_i}(\lambda f + (1-\lambda)q) \geq U(p)$, hence $\lambda f + (1-\lambda)q \in c_1(\{\lambda f + (1-\lambda)q, p\})$. We conclude that DM 1 with choice correspondence represented by (U, P) optimistically learns from DM 2.

Part 2. For the other direction, suppose DM 1 optimistically learns from DM 2. Notice that DM 1 satisfies *WARP*, and hence her choice is represented by the utility function of the maxmin model $(U_1, \{P\})$. This utility function is equal to

$$W(f) \;=\; \sup_{\lambda \in (0,1]} \Big(\frac{1}{\lambda}(Q_\lambda(f)) - (1-\lambda)U(q)\Big)$$

where

$$Q_\lambda(f) \;=\;$$

$$\sup\Big\{U_2(p) \;\Big|\; p \in H_0 : \exists f_i,...,f_N \in H, \sigma \in \triangle(\{1,...,N\}) : \sum_{i=1}^{N} \sigma_i f_i = \lambda f + (1-\lambda)q \;,\; f_i \in c_2(\{f_i, p\})\Big\}$$

Notice that by our calculations in Part 1, maxmin model $(U_2, \bigcap_{i=1,..,N} P_i)$ is represented by the same utility function. Therefore, these two model represent the same choice correspondence, hence U_2 is a positive affine transformation of U_1, and $P = \bigcap_{i=1,..,N} P_i$. ■

Appendix B.

Proof of Proposition 1

Part 1. Independence of *Framed Uncertainty*. Let $X = \{x, y, z\}$ and S be arbitrary. Consider

$$c(A) =$$

$$\left\{ f \in A \,\middle|\, \sum_{s\in S} f(s)(x) \geq \sum_{s\in S} g(s)(x) \ \forall g \in A \right\} \cup \left\{ f \in A \,\middle|\, \sum_{s\in S} f(s)(y) \geq \sum_{s\in S} g(s)(y) \ \forall g \in A \right\}$$

Notice that $c(\{x, 0.5x+0.5y\}) = \{x, 0.5x+0.5y\}$ and $c(\{x, 0.5x+0.5y, y\}) = \{x, y\}$, thus *C-β*, and, hence, *Framed Uncertainty* is violated. It is easy to see that *C-Independence*, *No-C-Hedging* and *Strict Monotonicity* are satisfied. Next, let the premise of the *Indirect Ambiguity Aversion* holds for acts h, f, g and menu A. Suppose that for some $i \in \{x, y\}$ we have $\sum_{s\in S} h(s)(i) \geq \sum_{s\in S} h'(s)(i)$ for all $h' \in A$. Then $\sum_{s\in S} h(s)(i) < \sum_{s\in S} f(s)(i)$ and $\sum_{s\in S} h(s)(i) < \sum_{s\in S} g(s)(i)$, therefore $\sum_{s\in S} h(s)(i) < \sum_{s\in S} (\lambda f(s)(i) + (1-\lambda) g(s)(i))$, hence $h \notin c(A \cup \{\lambda f + (1-\lambda) g\})$, and the *Indirect Ambiguity Aversion* holds. Next, let $f^k \in c(A^k)$ for $k = 1, 2, \ldots$, and $(A^k, f^k) \longrightarrow (A, f)$. Then at least for one $i \in \{x, y\}$ there exists a subsequence (A^{k_l}, f^{k_l}) such that $\sum_{s\in S} f^{k_l}(s)(i) \geq \sum_{s\in S} g(s)(i)$ for all $g \in A^{k_l}$. Since $(A^{k_l}, f^{k_l}) \longrightarrow (A, f)$, it follows that $\sum_{s\in S} f(s)(i) \geq \sum_{s\in S} g(s)(i)$ for all $g \in A$, hence $f \in c(A)$, and *Continuity* holds. Finally, $z \notin c(\{0.5x + 0.5y, z\})$, thus *C-Non-Degeneracy* holds.

Part 2. Independence of *C-Independence*. Let $X = \{x, y\}$, $S = \{1, 2\}$. Consider

$$c(A) = \arg\max_{f\in A} W(f) \quad , \quad \text{where} \quad W(f) = \frac{f(1)(x) + f(2)(x)}{1 + f(2)(x)}$$

Let $f = (0.4x + 0.6y, y)$ and $g = (y, x)$. Then $W(f) = 0.4 < 0.5 = W(g)$, hence $c(\{f, g\}) = \{g\}$. Next, let $p = (x, x)$, and consider $f' = 0.5f + 0.5p = (0.7x + 0.3y, 0.5x + 0.5y)$ and $g' = 0.5g + 0.5p = (0.5x + 0.5y, x)$. Then $W(f') = 1.2/1.5 = 4/5 > 3/4 = 1.5/2.0 = W(g')$, hence $c(0.5\{f, g\} + 0.5\{p\}) = \{f'\} = \{0.5f + 0.5p\}$, contradicting *C-Independence*. Next, $c(\cdot)$ has a continuous utility representation, thus *WARP*, and, hence, *Framed Ambiguity* holds, and also *Continuity* holds. Clearly, *C-Non-Degeneracy* holds: $W(y, y) = 0 < 1 = W(x, x)$. Next, since $W(p) \geq W(q)$ iff $p(x) \geq q(x)$, and $W(f) = 1 - \dfrac{1 - f(1)(x)}{1 + f(2)(x)}$, *Strict Monotonicity* holds as well. Finally, since $\dfrac{a}{b} \leq \dfrac{c}{d}$ implies $\dfrac{a}{b} \leq \dfrac{\lambda a + (1-\lambda)c}{\lambda b + (1-\lambda)d} \leq \dfrac{c}{d}$ for positive a, b, c, d and

$\lambda \in (0,1)$, it follows that $\min\{W(f), W(g)\} \leq W(\lambda f + (1-\lambda)g) \leq \max\{W(f), W(g)\}$, which imply *No-C-Hedging* and *Indirect Ambiguity Aversion*. Indeed, the premise of the former axiom implies $W(h) \geq \max\{W(f), W(p)\}$ for $h \in c(A)$, and the premise of the latter axiom implies $W(h) < \min\{W(f), W(g)\}$ whenever $h \in c(A)$.

Part 3. Independence of *No-C-Hedging*. Let $X = \{x, y\}$ and $S = \{1,2\}$. Then $H \simeq [0,1]^2$. Take some $0 < \underline{\mu} \leq \overline{\mu} \leq 1$ and consider

$$c(B) = \bigcup_{k \in [0,\infty)} \arg\max_{f \in B} W_k(f) \quad , \quad W_k(f) = \begin{cases} (1-\overline{\mu})(f(1)-f(2)) + f(2) & \text{if} \quad f(1) \geq f(2) \\ f(1) + \underline{\mu} \cdot \min\{(f(2)-f(1)), k\} & \text{if} \quad f(1) < f(2) \end{cases}$$

for any $B \in \mathcal{K}$. Take menu $A = \{(0.2, 0.3), (0.2, 1.0), p\}$, where $p = (0.2 + 0.1\underline{\mu}, 0.2 + 0.1\underline{\mu})$ is a constant act. Then $(0.2, 0.3) \in \arg\max_{f \in A} W_k(f)$ if and only if $k = 0.1$. Notice that $W_{0.1}(0.5 \cdot (0.2, 1.0) + 0.5 \cdot p) = W_{0.1}(0.2 + 0.05\underline{\mu}, 0.6 + 0.05\underline{\mu}) > W_{0.1}(0.2, 0.3)$. Therefore, $(0.2, 0.3) \in c(A)$, but $(0.2, 0.3) \notin c(A \cup \{0.5(0.2, 1.0) + 0.5p)\}\})$, violating *No-C-Hedging*.

It is easy to check that α, *Aizerman's Property*, *C-β*, *Strict Monotonicity* and *C-Non-Degeneracy* are satisfied for $c(\cdot)$. Notice that for arbitrary $k \in [0,\infty)$, $\lambda \in (0,1)$, act $f \in H = [0,1]^2$ and constant act $p = (a,a)$ we have $W_k(\lambda f + (1-\lambda)p) = \lambda W_{k/\lambda}(f) + (1-\lambda)a$, hence $W_k(\lambda f + (1-\lambda)p) \geq W_k(\lambda g + (1-\lambda)p) \iff W_{k/\lambda}(f) \geq W_{k/\lambda}(g)$. Since $\lambda : [0,\infty) \to [0,\infty)$ given by $\lambda(k) = k/\lambda$ is a bijection for $\lambda \in (0,1)$, we conclude that $c(\cdot)$ satisfies *C-Independence*. Next, $W_k(f) = \min\Big\{ (1-\overline{\mu})(f(1)-f(2)) + f(2),\ f(1) + \underline{\mu}(f(2)-f(1)),\ f(1) + \underline{\mu}k \Big\}$ is quazi-concave, thus $W_k(h) < W_k(f), W_k(g)$ implies $W_k(h) < W_k(\lambda f + (1-\lambda)g)$, hence *Indirect Ambiguity Aversion* holds for $c(\cdot)$. Finally, consider $(A^n, f^n) \longrightarrow (A, f)$ such that $f^n \in A^n$ for all $n = 1, 2, \ldots$. Notice that $W_k(h) = W_1(h)$ for all $k > 1$ and $h \in H$. Therefore, for all $n = 1, 2, \ldots$ there is $k(n) \in [0,1]$ such that $W_{k(n)}(f^n) \geq W_{k(n)}(h)$ for all $h \in A^n$. Since $[0,1]$ is compact, there is a convergent subsequence $k(n_m) \longrightarrow k \in [0,1]$. Next, since $A^{n_m} \longrightarrow A$, for any $g \in A$ there is $g^{n_m} \in A^{n_m}$ such that $g^{n_m} \longrightarrow g$. Hence, $W_{k(n_m)}(f^{n_m}) \geq W_{k(n_m)}(g^{n_m})$. Obviously, $W_k(h)$ is continuous w.r.t. $(k,h) \in [0,1] \times H$. Therefore, $W_k(f) \geq W_k(g)$. Since this holds for arbitrary $g \in A$, we conclude that $c(\cdot)$ satisfies *Continuity*.

Part 4. Independence of *Strict Monotonocity*. Let $X = \{x, y\}$, $S = \{1,2\}$, and

$$c(A) = \arg\max_{f \in A} W(f) \quad , \quad \text{where} \quad W(f) = 2f(1)(x) - f(2)(x)$$

Consider $f = (0.3x + 0.7y, 0.3x + 0.7y)$ and $g = (0.4x + 0.6y, 0.7x + 0.3y)$. Then $f(1) \notin c(\{f(1), g(1)\})$ and $f(2) \notin c(\{f(2), g(2)\})$, but $W(f) = 0.3 > 0.1 = W(g)$, hence $f \in c(\{f, g\})$, and *Strict Monotonicity* is violated. Since $c(\cdot)$ has a continuous utility representation, it satisfies *WARP*, and thus *Framed Uncertainty*, and also it satisfies *Continuity*. Next, $W(\lambda f + (1-\lambda)g) = \lambda W(f) + (1-\lambda)W(g)$, thus *C-Independence*, *No-C-Hedging*, *Indirect Ambiguity Aversion* hold. Finally, $W(x) = 1, W(y) = 0$, hence *C-Non-Degeneracy* holds as well.

Part 5. Independence of *Indirect Ambiguity Aversion*. Let $X = \{x, y\}$, $S = \{1, 2\}$, and

$$c(A) = \arg\max_{f \in A} W(f) \quad , \quad \text{where} \quad W(f) = \max\{f(1)(x), f(2)(x)\}$$

Let $h = (0.7x+0.3y, 0.7x+0.3y)$, and $A = \{h\}$. Notice that $h \notin c(\{h, (x, y)\})$ and $h \notin c(\{h, (y, x)\})$, but $h \in c(\{h, (0.5x+0.5y, 0.5x+0.5y)\})$, hence *Indirect Ambiguity Aversion* fails. Since $c(\cdot)$ has a continuous utility representation, it satisfies *Continuity* and *Framed Uncertainty*. Next, since $W(\lambda f + (1-\lambda)p) = \lambda W(f) + (1-\lambda)p(x)$, thus *C-Independence* and *No-C-Hedging* hold. Finally, $W(x) = 1 > 0 = W(y)$, thus *C-Non-Degeneracy* holds.

Part 6. Independence of *Continuity*. A choice corresondence that maximizes a lexicographic preference relation works as a standard example. Another example is a model that uses representation given by eq. (1) and eq.(2) with $X = \{x, y\}$, $S = \{1, 2\}$, $U(p) = p(x)$, $\mathcal{A} = \{\{\mu\} \subseteq \triangle S \mid 0 \leq \mu(1) < 1\}$, then $(x, y) \in c(\{(x, y), (\lambda x + (1-\lambda)y, \lambda x + (1-\lambda)y)\})$ for $\lambda \in [0, 1)$, but not for $\lambda = 1$, so *Continuity* fails. One can see that other axioms are satisfied.

Part 7. Independence of *C-Non-Degeneracy*. A trivial choice correspondence $c(A) = A$ for all $A \in \mathcal{K}$ satisfies all axioms except *C-Non-Degeneracy*. ■

Proofs of other statements

Lemma 27. *There is a choice correspondence $c(\cdot)$ that satisfies axioms 1-6 and violates axiom 7 (C-Non-Degeneracy) such that $c(A) \neq A$ for some $A \in \mathcal{K}$; moreover, $c(\{f, g\}) \neq \{f, g\}$ for some $f, g \in H$ and $c(A) = A$ for all $A \in \mathcal{K}_0$.*

Proof. Let $X = \{x, y\}$, $S = \{1, 2\}$, then $H \simeq [0, 1]^2$. Consider

$$c(A) = \{ f \in A \mid f(1) \geq f(2) \} \cup \{ f \in A \mid f(2) \geq g(2) \ \forall g \in A \}$$

Obviously, $c(A) \subseteq A$. Since $A \in \mathcal{K}$ is a compact set and function $u: H \to R$ given by $u(f) \equiv f(2)$ is continuous, there exists $f \in A$ such that $f(2) \geq g(2)$ for all $g \in A$. Thus, $c(\cdot)$ is a valid choice correspondence. Consider $f = (0, 0.5)$, $g = (0.1, 0.7)$ and $A = \{f, g\}$, then $c(A) = \{g\} \neq \{f, g\} = A$. Notice that $p(1) = p(2)$ for all $p \in H_0$, therefore $c(A) = A$ for all $A \in \mathcal{K}_0$, and in particular *C-Non-Degeneracy* is violated. It remains to check that other axioms are satisfied:

(α): Let $f \in c(A \cup B) \cap A$. If $f(1) \geq f(2)$, then $f \in c(A)$. If $f(1) < f(2)$, then $f(2) \geq g(2)$ for all $g \in A \cup B$, thus $f \in c(A)$.
C-β: Follows from $c(A) = A$ for all $A \in \mathcal{K}_0$
Aizerman's Property: Let $h \in c(A)$. If $h(1) \geq h(2)$, then $h \in c(A \cup \{f\})$. Otherwise $h(2) \geq g(2)$ for all $g \in A$. Then if $h(2) \geq f(2)$, then $h \in c(A \cup \{f\})$. Otherwise $f(2) > h(2) \geq g(2)$ for all $g \in A$, hence $f \in c(A \cup \{f\})$.
C-Independence: For $p \in H_0$ denote $f_\lambda = \lambda f + (1-\lambda)p$, etc. Notice that $p(1) = p(2)$. C-independence holds since for any $\lambda \in (0.1)$ we have: $f_\lambda(1) \geq f_\lambda(2) \iff f(1) \geq f(2)$ and $f_\lambda(2) \geq g_\lambda(2)\ \forall g_\lambda \in \lambda A + (1-\lambda)p \iff f_\lambda(2) \geq g_\lambda(2)\ \forall g \in A$.
No-C-Hedging: Let $h \in c(A)$ with $f, p \in A$. If $h(1) \geq h(2)$, then $h \in c(A \cup \{\lambda f + (1-\lambda)p\})$. Otherwise, $h(2) \geq g(2)$ for all $g \in A$, and in particular $h(2) \geq f(2), p(2)$. Thus, $h(2) \geq \lambda f(2) + (1-\lambda)p(2)$. Therefore, $h \in c(A \cup \{\lambda f + (1-\lambda)p\})$.
Strict Monotonicity: Since $c(\{p, q\}) = \{p, q\}$ for all $p, q \in H_0$, the premise of the axiom is never satisfied.
Indirect Ambiguity Aversion Consider $h \in A$. If $h \notin c(A)$, then, by proven condition α we have $h \notin c(A \cup \{\lambda f + (1-\lambda)g\})$. Suppose $h \in c(A)$. If $h(1) \geq h(2)$, then $h \in c(A \cup \{\lambda f + (1-\lambda)g\})$. Otherwise, $h(2) \geq h'(2)$ for all $h' \in A$. Let now $h \notin c(A \cup \{f\})$ and $h \notin A \cup \{g\}$. Then $h(2) < f(2), g(2)$. Therefore, $h(2) < \lambda f(2) + (1-\lambda)g(2)$. Given that $h(1) < h(2)$, it follows that $h \notin c(A \cup \{\lambda f + (1-\lambda)g\})$.
Continuity: Let $(A^n, f^n) \to (A, f)$ be such that $f^n \in c(A^n)$ for all $n = 1, 2, \ldots$. If $f(1) \geq f(2)$, then $f \in c(A)$. Otherwise $f(1) < f(2)$ and, since $f^n \to f$, then $f^n(1) < f^n(2)$ for large enough n. Therefore, $f^n(2) \geq g(2)$ for all $g \in A^n$. Notice that function $v: \mathcal{K} \to R$ given by $v(A) = \max_{g \in A} g(2)$ is continuous, thus function $u: \mathcal{K} \times H \to R$ given by $u(h, A) = h(2) - \max_{g \in A} g(2)$ is continuous. Hence, $f^n(2) - \max_{g \in A^n} g(2) = 0$ for all large enough n implies $f(2) - \max_{g \in A} g(2) = 0$ in the limit. It follows that $f(2) \geq g(2)$ for all $g \in A$, hence $f \in c(A)$. ■

Lemma 28. *Let $\mathcal{A} \subseteq \Pi$ be a non-empty family of sets of beliefs, then $(\mathcal{A})_{coh}$ given by eq. (3) is such that: (i) $\mathcal{A} \subseteq (\mathcal{A})_{coh}$, (ii) if a non-empty subcollection $\mathcal{C} \subseteq (\mathcal{A})_{coh}$ has a coherent intersection P, then $P \in (\mathcal{A})_{coh}$, and (iii) if $\mathcal{A}'$ is such that $\mathcal{A} \subseteq \mathcal{A}'$, and if a non-empty subcollection $\mathcal{C} \subseteq \mathcal{A}'$ has a non-empty intersection P, then $P \in \mathcal{A}'$, then $(\mathcal{A})_{coh} \subseteq \mathcal{A}'$.*

Proof. Statement (i) follows from $P \in \mathcal{A}$ being a trivial coherent intersection of $\{P\} \subseteq \mathcal{A}$. Statement (iii) follows from the definition of $(\mathcal{A})_{coh}$ and the fact that $\mathcal{C} \subseteq \mathcal{A}'$ for any non-empty subcollection $\mathcal{C} \subseteq \mathcal{A}$. To prove (ii), consider an arbitrary non-empty subcollection $\mathcal{C} \subseteq (\mathcal{A})_{coh}$ that has a coherent intersection P. Then for each $P' \in \mathcal{C}$ there exists $\mathcal{C}_{P'} \subseteq \mathcal{A}$ such that P' is a coherent intersection of $\mathcal{C}_{P'}$. Therefore, for any $z \in R^{|S|}$

$$I_P(z) \quad = \quad \max_{P' \in \mathcal{C}} I_{P'}(z) \quad = \quad \max_{P' \in \mathcal{C}} \max_{P'' \in \mathcal{C}_{P'}} I_{P''}(z) \quad = \quad \max_{P'' \in \left(\bigcup_{P' \in \mathcal{C}} \mathcal{C}_{P'}\right)} I_{P''}(z)$$

We conclude that P is a coherent intersection of $\bigcup_{P' \in \mathcal{C}} \mathcal{C}_{P'} \subseteq \mathcal{A}$, hence $P \in (\mathcal{A})_{coh}$. ■

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