Framing of Ambiguity

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Abstract

We develop and analyze a model of framing under ambiguity. *Frames* are circumstances, unobservable to the analyst, that shape the agent's perception of the relevant ambiguity. The analyst observes a choice correspondence that represents the set of possible choices under the various decision frames. We provide axioms that allow us to operationalize each frame as a set of priors, while the agent's utility index remains fixed. We show that the analyst can identify the unique minimum set of decision frames. One agent is *more consistent* than another if the former has a unique choice whenever the latter does. We characterize comparative consistency in terms of the model parameters and apply this result to characterize the aggregation of preferences that satisfy the Unanimity criterion. Finally, we characterize the behavior of agents who recognize that they are subject to different frames and learn by combining their frames into a single model.

Keywords: Framing Effects, Ambiguity, Multiple Priors, Aggregation of Preferences, Learning.

JEL Classifications: D01, D71, D80, D81, D90.

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1. Introduction

In this paper, we present a model of decision making under uncertainty with framing effects. More specifically, we analyze decision-makers who confront Knightian uncertainty and whose reasoning about this uncertainty is influenced by the manner in which the choice problem is presented. Each frame suggests a different assessment of the underlying uncertainty and therefore, may yield a different choice from a given set of options. Once the frame is fixed, the decision maker is a maxmin expected utility maximizer. The frame determines the set of probabilities over which expected utility is minimized.

Our motivation is the observation that decision makers are constrained by limited cognitive resources; they suffer from limited attention, limited memory or computational ability or a coarse understanding of contingencies. Such decision makers cannot identify, assess and integrate all available payoff-relevant information into unified decision procedure. In the absence of such a coherent procedure, the decision maker may be prone to mistakes and biases and may end up making different choices in seemingly identical situations.

To illustrate our framework, we consider a hypothetical student, Alice, preparing for an exam of unknown difficulty. She needs to decide whether to spend the day studying for the exam or work on her history paper. She mentions the exam to one of her friends who asks whether Alice thinks the exam will be *hard* or *not hard.* Alice realizes that if the exam is hard, she is likely to fail no matter how she spends the day. However, if the exam is not hard, an extra day of studying is likely to make the difference between success and failure. Hence, Alice decides to devote her day to studying for the exam.

Consider the following alternative scenario in which Alice, instead of discussing the exam with a friend, mentions it to her brother who states that in his opinion, the key is to assess whether the exam will be *easy* or *not easy.* Alice figures that if the exam is *not easy*, then there is a possibility of a failure whether she studies for the exam or not, and if the exam is *easy*, she will likely pass even if she spends the day on her history paper. Based on this reasoning, Alice concludes that she should devote her day to the history paper.

Note that each frame yields an incomplete specification of the underlying uncertainty. We interpret this incompleteness as ambiguity. With the friend's framing, Alice contemplates the events *hard* and *not hard* and hence these events become unambiguous while the events *easy* and *neither easy nor hard* are suppressed and remain ambiguous. With the brother's framing, Alice contemplates the events *easy* and *not easy* and hence these events become unambiguous while the events *hard* and *neither easy nor hard* are suppressed and remain ambiguous.

The analyst is unlikely to know whether Alice's framing of the problem was influenced by her friend or her sister; moreover, it is possible that Alice operates with a different set of states/events than the analyst. For the outside observer, such frame-susceptible agent behaves *as if* she has multiple ways to perceive the uncertainty. In this paper, we remain agnostic about the particular way how the unobservable framing affects the agent's perception of uncertainty; we focus, instead, on the analysis of the resulting observable choices.

The primitive of our *Framed Ambiguity* model is a choice correspondence that maps non-empty compact sets of Anscombe-Aumann acts to subsets of possible choices from those sets. Our axioms on this correspondence are equivalent, by Theorem 1, to the following representation:

$$\begin{aligned} c(A) &= \bigcup_i c_i(A) \\ c_i(A) &= \left\{ f \in A \;\middle|\; \min_{\mu \in P_i} \mathbb{E}_\mu U(f) \geq \min_{\mu \in P_i} \mathbb{E}_\mu U(g) \;\forall g \in A \right\} \end{aligned}$$

for all i. Hence, each frame i identifies a set of probability distributions P_i describing the uncertainty associated with that frame, while vNM expected utility function U remains the same for all frames. After adopting a particular frame, the agent becomes a Gilboa and Schmeidler (1989) maxmin expected utility maximizer.

To take into account the agent's susceptibility to framing, we require *Weak Axiom of Revealed Preference* (*WARP*) to hold only for choices among constant acts, when framing of uncertainty is irrelevant. For general menus, we relax *WARP* by retaining Sen's α (contraction consistency) but replacing Sen's β (expension consistency) with a weaker axiom, *Aizerman's Property*, which says that if an act added to the menu is not chosen, then acts chosen in the original menu are chosen in the enlarged menu as well.

Under framing, a decision maker exhibits "ambiguity-loving" behavior: she may choose one act under one frame, another act under another frame, but never choose a mixture of the two acts (Proposition 2). Our *Indirect Ambiguity Aversion* axiom demands that adding a convex combination of two acts to the menu should prevent a third act to be chosen whenever adding each of the two acts to the menu separately also prevents this third act to be chosen.

In Theorem 2, we show that the analyst can identify the agent's expected utility function up to a positive affine transformation and also a *unique minimum family of frames*. Moreover, we characterize the set of redundant frames—those that can be added to the minimum representation—as frames constructed from the minimum family by intersections of the corresponding sets of beliefs that satisfy a novel *Coherency* property. An intersection of sets in a linear space is *coherent* if the intersection operation commutes with projections on linear subspaces. If all sets of beliefs in a representation are singletons; that is, if the decision maker is a subjective expected utility maximizer with frame-dependent beliefs, then the representation is unique (Corollary 1).

We consider two criteria for comparing framed-ambiguity agents: decision maker 1 is *more decisive* than decision maker 2 if from each menu, decision maker 1 chooses a subset of acts chosen by decision maker 2. Decision maker 1 is *more consistent* than decision maker 2, if whenever decision maker 2 chooses a unique option from a menu, so does decision maker 1.

In Proposition 3, we show that one decision maker is more decisive than another if and only if the two have the same expected utility functions and the former's maximum family of frames is a subset of the latter's maximum family of frames. Hence, a more decisive agent is one who has fewer frames. Theorem 3 provides a characterization of the more consistent than relationship for decision makers with finitely many frames. Unlike greater decisiveness, greater consistency does not require fewer frames; rather, it imposes restrictions on the type of additional frames that a more consistent agent can have.

In Corollary 3, we apply Theorem 3 for the special case when one of the decision makers has a single frame to characterize the set of *Unanimous* rules (Crès et al. (2011)) that aggregate sets of beliefs of a group of Gilboa and Schmeidler (1989) agents who agree on utilities.

An agent who contemplates different frames and considers only priors that belong to all frames behaves optimistically: she may choose an act in comparison to a constant alternative even if there is no frame under which this choice is optimal. We call this type of behavior, novel in the literature to the best of our knowledge, "optimistic learning" and characterise it in Proposition 6.

1.1. Related Literature

Our paper contributes to the literature on framing, ambiguity and aggregation of beliefs. In this section, we introduce the related papers; in subsequent sections, we discuss the relation of our results to the literature.

Tversky and Kahneman (1981) introduced the notion of framing. In their formulation, framing introduces a benchmark that enables decision makers to identify some outcomes as *gains* and others as *losses*. In our formulation, a frame enables a decision maker to organize her reasoning about the uncertainty.

Framed ambiguity model extends Gilboa and Schmeidler (1989) maxmin model of ambiguity aversion. We enrich it by allowing for multiple sets of beliefs and interpreting each set of beliefs as a frame as in Salant and Rubinstein (2008)[1].

The Anscombe-Aumann framework in Heller (2012) is a special case of our model under two restrictions: all sets of beliefs are singletons, and the family of frames (singleton beliefs) is convex. Heller (2012) builds on Lehrer and Teper (2011), where the primitive is preference relation. In Section 4 of Lehrer and Teper (2011), the authors consider a "Bewley-type" generalization of their model: the decision maker prefers one act to another if and only if there exists at least one set of priors such that for every prior in this set, the first act is better than the second. The agent's behavior is consistent both with this model and our model if and only if it is consistent with the baseline model of Lehrer and Teper (2011)[2].

In Chandrasekher et al. (2022), the authors consider a decision maker whose perception of uncertainty is given by a family of sets of beliefs. They assume that the decision maker considers all sets of beliefs in the family, and chooses the most optimistic set of beliefs and the most pessimistic belief in a set to evaluate a given act. Thus, their agent respects *WARP* in contrast to ours.

Kopylov (2021) builds a model where an agent uses maxmin models with different sets of beliefs to evaluate acts. In his model, sets of beliefs are menu-dependent; moreover, each menu corresponds to exactly one set of beliefs. We, in contrast, assume that the decision maker can use all frames from the family to make choices from each menu. Thus, the two models are not nested.

Lu (2021) studies a random maxmin model. His primitive is a stochastic choice from menus consisting of ex-ante lotteries over Anscombe-Aumann acts[3]. In terms of his model, we assume less choice information available for the analyst: only the support of the distribution of choices from menus consisting of degenerate lotteries over acts is known.

[1] We also assume that the analyst does not observe a frame, but observes only a resulting choice correspondence, as in section 3 of Salant and Rubinstein (2008).

[2] The decision maker's behavior in Section 4 of Lehrer and Teper (2011) always satisfies the Independence axiom, but can violate Completeness. In our paper, in contrast, the decision maker constrained to choose from binary menus satisfies Completeness, since she always chooses at least one act from the menu, but violates Independence whenever there is a non-singleton set of priors.

[3] He considers also stochastic choices from menus of Anscombe-Aumann acts (i.e. degenerate lotteries over Anscombe-Aumann acts) in the Appendix of the paper (section A5).

Stoye (2011) characterizes a model of minimax regret. In his model, the agent chooses acts that minimize the worst-case—with respect to a set of priors and acts in the menu—expected regret. The agent's behavior can be represented by both Stoye (2011)'s model and our model if and only if the agent is a subjective expected utility maximizer.[4]

There is a theoretical literature that, in contrast to our setup, considers framing observable to the analyst. In Ahn and Ergin (2010), the authors assume that the analyst observes preference relations for various frames given by the partition the relevant state space and that the agent is an expected utility maximizer within each frame. Bourgeois-Gironde and Giraud (2009) assume that each action can be interpreted in terms of a frame-dependent Savage act, and allow the analyst to observe, in particular, comparisons between the same action framed differently. Their setup allows the agent to exhibit various attitudes towards ambiguity. In Caplin and Martin (2020), the authors model a frame as a mapping from agent's actions to consequences. The expected-utility maximizing agent knows only the disitribution of frames, but not the realized frame, while the analyst observes the latter as well.

Outside of the uncertainty framework, preference identification in the environment with framing is studied in Goldin and Reck (2020). Other decision-theoretic papers that consider violations of rationality in decision making under uncertainty include Ok et al. (2012), Galaabaatar and Karni (2013) and Hara et al. (2019).

Our comparative statics result contributes to the literature on aggregation of preferences under uncertainty. In Corollary 3, we consider aggregation of opinions of ambiguity averse experts that agree on utilities, but disagree on sets of beliefs. We discuss the relation of our results to Crès et al. (2011) and Hill (2011) in Section 4.4. Nascimento (2012) and Danan et al. (2016) are also concerned with aggregation of preferences under uncertainty.

1.2. Structure of the Paper

The rest of the paper is organized as follows. Section 2 introduces our setup, axioms and framed ambiguity model and provides characterization and identification results. Section 3 analyzes behavioral implications of agent's sensitivity to framing. Section 4 provides our comparative statics results and application to aggregation of preferences. Section 5 discusses the relevant experimental literature, our results and techniques. All proofs are given in the Appendix.

[4] Under Stoye (2011)'s *Ambiguity Aversion* axiom, our model reduces to Gilboa and Schmeidler (1989) model (Proposition 2), which reduces to subjective expected utility model under Stoye (2011)'s *Independence* axiom.

2. Model

We consider an Anscombe-Aumann setup with an arbitrary set of prizes X and finite set of states of the world S with $|S| \geq 2$. An act $f: S \to \triangle X$ is a mapping from the state space to the set $\triangle X$ of finite-support probabilities (simple lotteries) on X. Denote the set of acts by H with typical elements $f, g, h, \ldots$ and endow it with the suprenum metric d induced by the suprenum metric on the set of simple lotteries. The set of acts H is a mixture space with mixture operation defined via $(\lambda f + (1-\lambda)g)(s) = \lambda f(s) + (1-\lambda)g(s)$. A constant act is an act that gives the same consequences in all states: $f(s) = p \; \forall s \in S$. We denote such acts by the corresponding lotteries $p, q, r, \ldots$ whenever it does not cause confusion. The set of constant acts is $H_0 \subset H$.

The decision maker (DM, she) chooses acts from menus $A, B, C, \ldots$ A menu is a non-empty compact subset of H such that the set of prizes $\{x \in X \,|\, \exists f \in A, s \in S : f(s)(x) > 0\}$—that is, the set of prizes that have positive probability to appear in some state for some act in the menu—is finite[5]. We denote the set of all menus by $\mathscr{K}$ and endow it with the Hausdorff metric ρ induced by metric d on H. A mixture between a menu and an act is defined via $\lambda A + (1-\lambda)g = \{\lambda f + (1-\lambda)g \mid f \in A\}$; note that this mixture is also a menu. The set of menus consisting of constant acts is $\mathscr{K}_0 \subset \mathscr{K}$.

2.1. Choice Correspondence

The analyst observes a *choice correspondence*[6] $c: \mathscr{K} \to 2^H$ such that $\varnothing \neq c(B) \subseteq B$ for any $B \in \mathscr{K}$. We interpret $c(B)$ as a subset of acts that is chosen from menu B by the DM. We make the following implicit assumptions:

(i) DM's perception of uncertainty could be influenced by the details of the decision problem unobserved by the analyst that do not provide DM with additional information about the state of the world. We call the collection of such details *a frame*.

(ii) An analyst is able to recover choice correspondence from the data by observing repeated choices from various menus. Inconsistency of choices emerges only either because the framing of the problem varies or because the agent is indifferent between alternatives.

(iii) The collection of frames that DM experiences is invariant across menus.

The first assumption is the premise of the project, as we are interested in the class of choice

[5]Absent the last requirement, a subjective expected utility maximizer may have empty choice from some menus.

[6]Modern literature discusses inference of choice correspondences from the stochastic choice data: Ok and Tserenjigmid (2019), Balakrishnan et al. (2022).

correspondences that emerge as a result of the *ambiguity framing*. Assumption (ii) in particular says that *no learning* occurs: notwithstanding the fact that DM faces multiple frames, she is unable to connect her previous analysis to the current situation. Assumption (iii) says that we don't consider menu-dependent frames. Our final assumption is:

(iv) Decision maker is cautious toward the uncertainty that she can envision.

Assumption (iv) is common for models involving Knightian uncertainty. With framing, a new motivation to use caution emerges: DM might think that the uncertainty relevant for the choice problem is framed in a way that erroneously inflates values of some acts relative to others—emphasizing priors that favor these acts. The worst case scenario evaluation can be a good approach to counteract this type of framing.

2.2. Axioms

We introduce behavioral axioms on the choice correspondence in the spirit of the discussed above implicit assumptions (i)-(iv). Recall that *Weak Axiom of Revealed Preference (WARP)*

WARP: $\forall A,B \in \mathscr{K} \quad c(A) \cap B \neq \varnothing \implies c(B) \cap A \subseteq c(A)$

is equivalent to conditions α and β taken together:

α: $\quad \forall A,B \in \mathscr{K} \quad c(A \cup B) \cap A \subseteq c(A)$

β: $\quad \forall A,B \in \mathscr{K} \quad c(A \cup B) \cap A \neq \varnothing \implies c(A) \subseteq c(A \cup B)$

Our first axiom relaxes *WARP* on the domain of menus that involve non-constant acts.

AXIOM 1 *(Framed Uncertainty):*

1.1 *(α)*: $\quad \forall A,B \in \mathscr{K} \quad c(A \cup B) \cap A \subseteq c(A)$

1.2 *(Aizerman's Property)*: $\quad \forall A \in \mathscr{K} \quad \forall f \in H \quad f \notin c(A \cup \{f\}) \implies c(A) \subseteq c(A \cup \{f\})$

1.3 *(C-β)*: $\quad \forall A,B \in \mathscr{K}_0 \quad c(A \cup B) \cap A \neq \varnothing \implies c(A) \subseteq c(A \cup B)$

If there is some framing under which an act is the best in the menu, it is the best in the subset of this menu under the same framing; this justifies α. *Aizerman's Property* relaxes β[7]; it says that adding to the menu an act that will not be chosen does not prevent other acts to be chosen. Since we analyze framing that influences only the perception of uncertainty, we require *WARP* to hold on the domain of menus consisting of constant acts, hence *C-β* should hold.

[7]Let β hold. If $f \notin c(A \cup \{f\})$, then $c(A \cup B) \cap A \neq \varnothing$ for $B = \{f\}$; hence, $c(A) \subseteq c(A \cup B) = c(A \cup \{f\})$ by β.

We generalize Gilboa and Schmeidler (1989)'s c-independence axiom to the choice setting:

AXIOM 2 *(C-Independence):* $\forall A \in \mathcal{K}\ \forall p \in H_0\ \forall \lambda \in (0,1)\ \ c(\lambda A + (1-\lambda)p) = \lambda c(A) + (1-\lambda)p$

C-Independence says that uncertainty regarding $s \in S$ does not matter for evaluation of constant acts, and that DM is able to factorize Knightian uncertainty and objective uncertainty regarding the payoff given by an act. When β is relaxed, *C-Independence* does not fully capture this intuition[8], and we add the following property:

AXIOM 3 *(No C-Hedging):*
$\forall A \in \mathcal{K}\ \forall f \in H\ \forall p \in H_0\ \forall \lambda \in (0,1)\ \ f, p \in A \implies c(A) \subseteq c(A \cup \{\lambda f + (1-\lambda)p\})$

To get the intuition behind *No C-Hedging*, suppose that in all circumstances, act $\lambda f + (1-\lambda)p$ seems to be so attractive that DM cannot choose h from $A \cup \{\lambda f + (1-\lambda)p\}$. Irrespectively of DM's perception of uncertainty, either f or p should be at least as attractive as $\lambda f + (1-\lambda)p$, since constant act p cannot effectively hedge against bad outcomes in f. Hence, the presence of both f and p in A would not allow DM to choose h from A under any circumstances as well.

AXIOM 4 *(Strict Monotonicity):* $\forall f, g \in H\ \ g(s) \notin c(\{f(s), g(s)\})\ \forall s \in S \implies g \notin c(\{f, g\})$

Strict Monotonicity says that DM chooses consequentially: if she does not choose $g(s)$ versus $f(s)$ for any contingency s, she never chooses g versus f irrespectively of the perceived uncertainty.

Our last structural axiom formalizes assumption (iv):

AXIOM 5 *(Indirect Ambiguity Aversion):* $\forall A \in \mathcal{K}\ \ \forall h \in A\ \ \forall f, g \in H\ \ \forall \lambda \in (0,1)$
$h \notin c(A \cup \{f\})$ and $h \notin c(A \cup \{g\}) \implies h \notin c(A \cup \{\lambda f + (1-\lambda)g\})$

If DM is averse to ambiguity within the frame, she always considers act $\lambda f + (1-\lambda)g$ to be as good as at least one of acts f or g. If adding either f or g to the menu does not allow h to be chosen under any frame, so should do a mixture of f and g.

Finally, we introduce our technical axioms[9]. We endow $\mathcal{K} \times H$ with the suprenum metric.

AXIOM 6 *(Continuity):* $\{\, (A, f) \in \mathcal{K} \times H \mid f \in c(A)\,\}$ *is closed*

AXIOM 7 *(C-Non-Degeneracy):* $\exists p, q \in H_0$ *such that* $\{p\} = c(\{p, q\})$

[8] See Proposition 1 on page 9 and its proof in the Appendix.

[9] We use *C-Non-Degeneracy* axiom instead of weaker statements $f \notin c(\{f, g\})$ for some $f, g \in H$, or $c(A) \neq A$ for some $A \in \mathcal{K}$ because there are models where the agent does not discriminate between constant acts ($c(A) = A$ for all $A \in \mathcal{K}_0$), but exhibits a non-degenerate choice—in particular, $f \notin c(\{f, g\})$ for some $f, g \in H$—that satisfies axioms 1–6 (Lemma 27). Although such choice behavior is intriguing, it is not the focus of this paper.

2.3. Representation

We denote by Π the set of all non-empty compact and convex sets of beliefs $P \subseteq \triangle S$. We endow Π with the Hausdorff metric. When we talk about a set of beliefs P, we assume that $P \in \Pi$.

DEFINITION 1. *A Framed Ambiguity model is a pair $(U, \mathscr{A})$, where $U : \triangle X \to R$ is a non-degenerate vNM expected utility function, and $\mathscr{A} \subseteq \Pi$ is a non-empty closed family of non-empty compact and convex sets of beliefs $P \in \Pi$.*

DEFINITION 2. *A framed ambiguity model $(U, \mathscr{A})$ represents choice correspondence $c(\cdot)$ if*

$$c(B) = \bigcup_{P \in \mathscr{A}} \arg\max_{f \in B} W_P(f) \qquad \forall B \in \mathscr{K} \tag{1}$$

where

$$W_P(f) = \min_{\mu \in P} \sum_{s \in S} \mu(s) U(f(s)) \tag{2}$$

Note that each framed ambiguity model represents a choice correspondence given by eq. (1), since $c(B) \subseteq B$, and $arg\ max$ in eq. (1) is non-empty for each $P \in \mathscr{A}$. The function $W_P : H \to \mathbb{R}$ is a utility function of the Gilboa and Schmeidler (1989) maxmin model with vNM expected utility U and set of beliefs P. An act belongs to $c(B)$ whenever it maximizes menu B with respect to at least one of utility functions $\{W_P\}_{P \in \mathscr{A}}$. Thus, a framed ambiguity model operationalizes frames via sets of beliefs $P \in \Pi$; the family $\mathscr{A}$ is the set of frames, and each of them could be used as a rational for choosing $f \in c(B)$. We maintain an assumption that a frame is an unobserved property of a decision problem and not a property of an individual act. Thus, an act is chosen if it is the best under some framing, when all acts in the menu are evaluated under this framing.

Our first main result is a characterization theorem.

Theorem 1. *A choice correspondence has a framed ambiguity representation if and only if Axioms 1–7 hold.*

The axioms *Framed Uncertainty* (in particular, α and *C-β*), *C-Independence, Continuity* and *C-Non-Degeneracy* imply the existence of the (unique up to a positive affine transformation) non-degenerate vNM expected utility function U that represents the DM's choice on the domain of constant acts. The rest of the axioms guarantee that the DM's choice over general menus is represented by a framed ambiguity model. Conversely, a choice correspondence induced by a framed ambiguity model satisfies axioms 1–7. Moreover, there are no redundant axioms:

Proposition 1. *Let*[10] $|X| > 2$, *and* S *be arbitrary, then for any axiom* $Ai \in \{$*Axiom 1, ..., Axiom 7*$\}$, *there is a choice correspondence* c_i *such that* Ai *fails, and other axioms hold.*

Consider the following example that sheds light on our identification result. Let $X = \{x, y\}$ and $S = \{1,2\}$. Agents 1 and 2 have the same vNM expected utility function $U_1(p) = U_2(p) = p(x)$ for $p \in \triangle X$. Denote by $\mu = Pr(s = 1)$, and let $\mu_1 < \mu_2 < \mu_3 < \mu_4$. The first agent has two frames: $\mathscr{A}_1 = \{P, Q\}$, where $P = conv(\{\mu_1, \mu_3\})$, $Q = conv(\{\mu_2, \mu_4\})$. The second agent has the same frames and also their intersection: $\mathscr{A}_2 = \{P, Q, P \cap Q\}$, $P \cap Q = conv(\{\mu_2, \mu_3\})$. Thus, if $f(1)(x) \geq f(2)(x)$, act f is evaluated according to prior μ_2 under frames $P \cap Q$, Q, and μ_1 under frame P. Similarly, if $f(1)(x) < f(2)(x)$, f is evaluated according to μ_3 under frames $P \cap Q$, P, and μ_4 under frame Q. Hence, $W_{P\cap Q}(f) = \max\{W_P(f), W_Q(f)\}$ for any f. Thus, if f maximizes $W_{P\cap Q}(\cdot)$ in a menu, f also maximizes at least one of $W_P(\cdot)$ or $W_Q(\cdot)$. It follows that both agents' choice behaviour is captured by the same correspondence $c(\cdot)$, and the analyst cannot differentiate the agents.

The multiplicity of representations in this example occurs because $W_{P\cap Q}(f) = \max\{W_P(f), W_Q(f)\}$ for any act f. Our results show that this is the only reason for multiplicity of representations (aside from the transformations of the vNM expected utility function). When $|S| = 2$, any closed collection $\mathscr{C}$ of sets of beliefs with non-empty intersection has the property that $W_{\bigcap_{P\in\mathscr{C}} P}(f) = \max_{P\in\mathscr{C}} W_P(f)$ for all f, thus the frame $\bigcap_{P\in\mathscr{C}} P$ can be added to or deleted from the representation without altering the induced choice correspondence. For arbitrary S, this is true if and only if the collection of frames satisfy the following *intersection-coherency* property[11] (Lemma 11):

DEFINITION 3. *A non-empty closed collection* $\mathscr{C}$ *of non-empty convex compact sets in* $\triangle S \subset \mathbb{R}^S$ *is intersection-coherent if for any linear subspace* $\mathbb{T}$ *of* $\mathbb{R}^S$,

$$\bigcap_{P\in\mathscr{C}} proj_{\mathbb{T}} P \;=\; proj_{\mathbb{T}}\Big(\bigcap_{P\in\mathscr{C}} P\Big) \;\neq\; \varnothing.$$

If collection $\mathscr{C}$ *is intersection-coherent, we say that* $\bigcap_{P\in\mathscr{C}} P$ *is a coherent intersection of* $\mathscr{C}$.

Note that a singleton collection $\{P\}$ is intersection-coherent, and its coherent intersection is P.

Theorem 2. *If* $c(\cdot)$ *satisfy axioms 1–7, then there is a unique minimum family of frames* $\mathscr{A} \subseteq \Pi$, *a unique maximum family of frames* $\mathscr{B} \subseteq \Pi$ *and vNM expected utility function* U *such that:*

(i) $(U', \mathscr{A}')$ *represents* $c(\cdot)$ *if and only if* U' *is a positive affine transformation of* U, *and* $\mathscr{A} \subseteq \mathscr{A}' \subseteq \mathscr{B}$;

(ii) $P \in \mathscr{B}$ *if and only if* P *is a coherent intersection of a collection of frames* $\mathscr{C} \subseteq \mathscr{A}$.

[10] If $|X| = 2$ the examples given in the proof of Proposition 1 go through as well, except one that shows the independence of $A1$; the question if A1 is independent of A2-A7 in case when $|X| = 2$ is open.

[11] The definition can be applied to arbitrary sets in arbitrary linear spaces without change.

Theorem 2 tells us that, although we cannot exclude some multiplicity of representations, there is a unique minimum collection of frames that provides a framed ambiguity representation of choice correspondence $c(\cdot)$ that satisfies our axioms. Coherent intersections of sub-collections of frames in the minimum representation—and only such frames—can be added to the representation without altering the induced choice correspondence.

To get the idea behind the identification result, consider some act f that is not the worst or the best[12]. Call menu D maximal for act f if $f \in c(D)$, but $f \notin c(D')$ when $D \subsetneq D'$. If DM's choices satisfy *WARP*, there exists exactly one such menu—the menu consisting of all acts that are weakly worse than f; that is, the lower counter set of f. With multiple frames, there are multiple maximal menus for an act, and each maxmimal menu corresponds to a lower contour set of f according to the Gilboa and Schmeidler (1989) maxmin model with some set of beliefs P. The family of sets of beliefs $\mathscr{A}$ recovered from the choice correspondence by this way is the minimum family of frames in the representation.

Since the intersection of two different singleton sets of beliefs is empty, Theorem 2 implies that in the special case when DM is a subjective expected utility maximizer with frame-dependent belief, $\mathscr{A}$ is a unique collection of frames (singleton beliefs) in the representation of $c(\cdot)$.

Corollary 1. *Let $(U, \mathscr{A})$ represent $c(\cdot)$. If all frames $P \in \mathscr{A}$ are singletons, then the representation is unique up to a positive affine transformation of U.*

2.4. Identification of Multi-Multiple Prior Models in the Literature

In this section, we discuss multi-multiple prior models in the literature and the extend to which the beliefs are identified in these models in comparison with the framed ambiguity model.

In Chandrasekher et al. (2022), the set of *relevant priors*—the convex hull of the priors that are used by DM to evaluate acts—is unique for the representation. This is true for our model as well: the set of relevant priors is a convex hull of the union of all frames in the minimum family[13]. Our model also exhibits a unique minimum family of sets of beliefs. Chandrasekher et al. (2022)'s primitive is a rational (complete and transitive) preference relation; observing rational (satisfying *WARP*) choices from non-binary menus would not benefit the analyst in their model. In contrast, in our model, the analyst benefits from observing choices from larger menus (Section 4.3).

[12]That is, f chosen in some non-singleton menu, and also not chosen in some menu.
[13]This follows from Theorem 2.

In Kopylov (2021), the analyst can identify the projections of sets of beliefs that account for the ambiguity relevant for given decision problems. In our model, the analyst can identify all priors in each set of beliefs completely, but some redundant sets of beliefs may occur.

Lu (2021) shows that the distribution of sets of beliefs is uniquely identified if the analyst knows either frequencies of choices from binary menus consisting of an ex-ante lottery over Anscombe-Aumann acts and a constant act, or frequencies of choices from arbitrary menus of Anscombe-Aumann acts (Theorem 7 in his paper). Lu (2021)'s model makes an implicit assumption that the process of assignment of a menu is independent of the process governing the choice of the set of beliefs that DM uses. Our model, in contrast, can accommodate a correlation between a menu and a set of beliefs as long as it is not perfect—we operate only with the choice correspondence $c(\cdot)$ that can be interpreted as the support of the distribution of choices for each menu.

In Lehrer and Teper (2011), the analyst identifies a convex family of singleton beliefs uniquely from the binary relation and similarly, in Heller (2012), the analyst identifies a convex family of singleton beliefs uniquely from the choice correspondence. Our Corollary 1 shows that a non-convex family of singleton beliefs is also identified uniquely from the choice correspondence. Our model, even constrained to singleton beliefs, admits a richer behavior: let DM have two frames: $P_1 = \{\mu_1\}$ that favors f to $0.5f + 0.5g$ to g, and $P_2 = \{\mu_2\}$ that favors g to $0.5f + 0.5g$ to f. Such DM chooses f and g, but not a compromise alternative $0.5f + 0.5g$ from the menu $\{f, 0.5f + 0.5g, g\}$. In contrast, DM in Heller (2012) should necessary choose a compromise alternative $0.5f + 0.5g$ if she chooses f and g, since this mixture is the best according to one of the intermediate beliefs $\lambda\mu_1 + (1-\lambda)\mu_2$ for some $\lambda \in (0,1)$.

3. Frame Sensitivity

In this section, we study the behavioral differences between agents who are *sensitive* to the framing of a choice problem—in the sense that they may change their decision depending on the framing of the decision problem—and agents who are not sensitive to the framing. We show that if framed-ambiguity agent respects one of the considered below axioms, then she respects them all and she is not sensitive to framing.

Following the analysis of Sen (1971), consider the *Revealed Preference* relation $\succeq \subseteq H^2$ defined by $f \succeq g \iff \exists A \in \mathscr{K} : g \in A, f \in c(A)$. Note that when α holds—in particular, in our model—we can equivalently define $\succeq$ as $f \succeq g \Longleftrightarrow f \in c(\{f,g\})$[14].

Revealed Preference Rationality: Revealed preference $\succeq$ is complete and transitive.

Next, consider the following *Property* γ from Sen (1971):

γ : $\forall \mathscr{D} : \varnothing \neq \mathscr{D} \subseteq \mathscr{K},\ \bigcup_{B \in \mathscr{D}} B \in \mathscr{K} \quad \bigcap_{B \in \mathscr{D}} c(B) \subseteq c(\bigcup_{B \in \mathscr{D}} B)$

Property γ says that if some alternative (act) is chosen in each of the sets B in collection $\mathscr{D}$, it must be chosen in their union as well. Sen (1971) shows that properties α and γ are equivalent to the following property:

Normality: $\forall A \in \mathscr{K} \quad c(A) = \{f \in A \mid f \succeq g\ \forall g \in A\}$, where $f \succeq g$ iff $\exists A \in \mathscr{K} : g \in A, f \in c(A)$

Normality says that the agent chooses from a menu by maximizing the revealed preference relation. As pointed out by Heller (2012), α, γ and *Aizerman's Property* do not imply *WARP*[15].

To characterize the minmax regret model, Stoye (2011) considers the following axiom:

Ambiguity Aversion: $\forall f, g \in H, \lambda \in [0,1]\ \forall A \in \mathscr{K} :\ \{g, f, \lambda f + (1-\lambda)g\} \subseteq A$
$f, g \in c(A) \implies \lambda f + (1-\lambda)g \in c(A)$

Ambiguity Aversion says that if the agent chooses acts f and g, she should choose act $\lambda f + (1-\lambda)g$, provided that all these acts are in the menu. This argument does not work in a model with frames: f can be chosen under frame 1, g under frame 2, but $\lambda f + (1-\lambda)g$ is not chosen under frames 1 and 2. When α holds, *Ambiguity Aversion* is implied by the following property:

Direct Ambiguity Aversion: $\forall A \in \mathscr{K} \quad \forall f, g \in H \quad \forall \lambda \in (0,1)$
$f \in c(A \cup \{f\})$ and $g \in c(A \cup \{g\}) \implies \lambda f + (1-\lambda)g \in c(A \cup \{\lambda f + (1-\lambda)g\})$

Finally, to clarify the role of *No-C-Hedging* axiom in our model, consider its alternative:

Pairwise No C-Hedging: $\forall f, h \in H,\ \forall p \in H_0$
$h \in c(\{h, f\})$ and $h \in c(\{h, p\}) \implies h \in c(\{h, \lambda f + (1-\lambda)p\})$

Pairwise No C-Hedging says that if, in binary comparisons with act f and constant act p, act h is chosen, then h must be chosen in a binary comparison with a mixture between acts f and p—intuitively, because this mixture cannot be strictly more attractive than both acts f and p. Again, this argument does not work in a model with frames.

[14]Note that even if we demand *Revealed Preference Rationality*, α and *Aizerman's Property* to hold together, this does not imply *WARP* without other axioms. For instance, it can be that in each of the three pairs of f, g, h, both alternatives are chosen, but $c(\{f, g, h\}) = \{f, g\}$.

[15]These three axioms together are equivalent to axiom *WARNI* from Eliaz and Ok (2006).

Proposition 2. *Let $c(\cdot)$ have framed ambiguity representation $(U, \mathscr{A})$. Then:*

(i) Either $|\mathscr{A}| = 1$ and properties β, WARP, Revealed Preference Rationality, γ, Normality, Ambiguity Aversion, Direct Ambiguity Aversion, Pairwise No-C-Hedging hold for $c(\cdot)$;

(ii) Or $|\mathscr{A}| > 1$ and each of the properties β, WARP, Revealed Preference Rationality, γ, Normality, Ambiguity Aversion, Direct Ambiguity Aversion, Pairwise No-C-Hedging is violated for $c(\cdot)$.

Statement (i) of Proposition 2 is more straightforward: when $|\mathscr{A}| = 1$, the framed ambiguity model reduces to the maxmin model of Gilboa and Schmeidler (1989) defined for a choice correspondence. Accordingly, the rationality axioms β, *WARP*, *Revealed Preference Rationality*, γ, *Normality* hold, and axioms *Ambiguity Aversion*, *Direct Ambiguity Aversion*, *Pairwise No-C-Hedging* that represent (rational) DM's attitude toward Knightian and objective uncertainty hold as well.

Statement (ii) of Proposition 2 is more surprising: it says that if there is some ambiguity framing, the DM's choice behavior violates all of the axioms discussed in Proposition 2. In particular, the DM's behavior violates *Normality* which is assumed for most of the analysis of choice with frames in Salant and Rubinstein (2008), and it violates *Ambiguity Aversion* which holds for the decision-maker who violates *WARP* in Stoye (2011).

If the analyst mistakenly considers the agent's behavior to be not frame-sensitive, observing violations of *Ambiguity Aversion* or *Direct Ambiguity Aversion*, the analyst can mistakenly conclude that the agent likes Knightian uncertainty, and observing a violation of *Pairwise No-C-Hedging*, the analyst can mistakenly conclude that the agent exhibits non-linear preference toward the objective uncertainty.

Finally, by checking *Revealed Preference Rationality* or *Pairwise No-C-Hedging*, the analyst can verify if the agent is frame-sensitive by observing only choices from binary menus.

4. Connection of Frames and Consistency of Choices

In this section, we analyze framed-ambiguity agents who differ in their ability to recognize that their perception of uncertainty is subject to framing. Consider an agent (DM 2, he) who is uncertain regarding the severity of the Global Warming[16]. One day, he listens to an expert who says that the probability p of Global Warming being severe is between 20 and 60 percents; the agent then acts—for instance, trades assets—based on estimate $p \in [0.2, 0.6]$. On a different day,

[16] We follow Crès et al. (2011) using Global Warming as a motivating example.

he listens to another expert who says that the relevant probability is between 50 and 75 percents; this time, the agent acts based on estimate $p \in [0.5, 0.75]$.

Another agent (DM 1, she) also has access to both estimates, but, in contrast to DM 2, she tries to connect the two points of view and come up with a more coherent picture of the relevant uncertainty. One—extreme—way to do this is to pick one of the experts and trust this expert in any circumstances, disregarding the other expert's opinion. In Proposition 3, we show that such agent is *more decisive* in comparison to DM 2: her choice correspondence is a subset of his. Moreover, this type of behavior characterizes comparative decisiveness.

The agent may also try to use both estimates. Unable to come up with a Bayesian framework to connect the two opinions, she may still use reasonable approaches. First, she may admit all possibilities, and estimate that the Global Warming is severe with probability between 20 and 75 percent; that is, $p \in [0.2, 0.75] = [0.2, 0.6] \cup [0.5, 0.75]$[17]. Second, she may consider only probabilities that are consistent with both estimates; this way, $p \in [0.5, 0.6] = [0.2, 0.6] \cap [0.5, 0.75]$. Third, she may assign each expert a weight[18]—for instance, 2/3 for the first expert and 1/3 for the second expert—and find a compromise range of probabilities: $p \in [0.3, 0.65] = \frac{2}{3} \cdot [0.2, 0.6] + \frac{1}{3} \cdot [0.5, 0.75]$.

The way the agent connects the expert's opinions may be itself subject to framing: one day, she uses estimate $[0.2, 0.75]$, the other day, she uses the range $[0.5, 0.6]$, while on a third day, she is inclined to use a compromise estimate $[0.3, 0.65]$. Moreover, the agent may use any of these "combined" estimates as a new "opinion". For instance, she may come up with estimate $p \in [0.3, 0.6] = [0.2, 0.6] \cap [0.3, 0.65] = [0.2, 0.6] \cap \left(\frac{2}{3} \cdot [0.2, 0.6] + \frac{1}{3} \cdot [0.5, 0.75]\right)$. In Theorem 3, we show that such agent is *more consistent* than DM 2: she has a singleton choice whenever DM 2 does. Moreover, the three considered ways of connecting/aggregating frames (experts' opinions) characterise comparative consistency—with a nuance that for non-binary state space, only coherent intersections of sets of beliefs are used.

Below, we formalize the ideas introduced above and characterise the (unobserved) aggregation of frames in terms of the (observed) comparative choice behavior of agents.

4.1. Decisiveness

We start by considering a natural decisiveness relation.

[17]When frames are singleton beliefs, this aggregation idea corresponds to *Default to Certainty* in Gilboa et al. (2010), and for general sets of beliefs it is analyzed in Crès et al. (2011) and Hill (2011)

[18]This way of aggregation of sets of beliefs is also considered in Crès et al. (2011) and Hill (2011).

DEFINITION 4. *DM 1 with choice correspondence $c_1(\cdot)$ is more decisive than DM 2 with choice correspondence $c_2(\cdot)$ if $c_1 \subseteq c_2$.*

Thus, we say that DM 1 is more decisive than DM 2 if she chooses among a subset of alternatives that could be chosen by DM 2. Let $(\mathscr{A}_2)_{coh}$ be the closure of the (non-empty and topologically closed) family of sets of beliefs $\mathscr{A}_2$ with respect to the operation of coherent intersection of frames—that is, $(\mathscr{A}_2)_{coh}$ is the minimum family of sets of beliefs that includes $\mathscr{A}_2$ and is closed under taking coherent intersections of its sub-collections.

Theorem 2 and Lemma 28 in the Appendix imply that $(\mathscr{A}_2)_{coh}$ exists, and it is the maximum family of frames that represents the choice correspondence induced by the family of frames $\mathscr{A}_2$ and a common vNM expected utility function.

Proposition 3. *Let $c_1(\cdot)$ and $c_2(\cdot)$ be represented by $(U_1, \mathscr{A}_1)$ and $(U_2, \mathscr{A}_2)$. Then DM 1 is more decisive than DM 2 if and only if U_1 is a positive affine transformation of U_2 and $\mathscr{A}_1 \subseteq (\mathscr{A}_2)_{coh}$.*

Thus, Proposition 3 tells us that a more decisive agent is one who starts with the maximum collection of frames that the other agent may have and drops some of those.

4.2. Consistency

We proceed with a less demanding notion, which compares instances in which agents make the same choice consistently.

DEFINITION 5. *DM 1 with choice correspondence $c_1(\cdot)$ is more consistent than DM 2 with choice correspondence $c_2(\cdot)$ if for all $A \in \mathscr{K}$ $\quad |c_2(A)| = 1 \implies |c_1(A)| = 1$.*

If DM 1 is more decisive than DM 2, then $c_2(A) = \{f\} \implies c_1(A) = \{f\}$, hence DM 1 is also more consistent than DM 2[19]. Let us now formally define the aggregation concepts discussed in the Global Warming example.

DEFINITION 6. *Given a non-empty closed collection of frames $\mathscr{C}$, its convex union is a set of beliefs P given by*

$$P \;=\; conv\left(\bigcup_{P' \in \mathscr{C}} P'\right) \tag{3}$$

Definition 6 describes the aggregation approach in which the agent contemplates a subset of frames $\mathscr{C}$ at a moment and think that any prior in any set of beliefs is possible; the convex hull

[19]Note also that comparative consistency and comparative decisiveness are reflexive and transitive binary relations on the set of framed-ambiguity agents. The latter relation is antisymmetric, but the former is not.

is taken because non-convex sets of beliefs—as oppose to non-convex sets of frames—are not identifiable in our model; this is true as well for the Gilboa and Schmeidler (1989) maxmin model.

The notion of coherent intersection of frames is introduced in Section 2 (Definition 3). Our next definition formalizes the idea of a weighted average of various opinions.

DEFINITION 7. *Given a non-empty finite collection of frames $\mathscr{C} = \{P_1, \ldots, P_N\}$, its convex combination with respect to weights $\lambda \in \triangle(\{1, \ldots, N\})$ is a set of beliefs P given by*

$$P = \sum_{i=1}^{N} \lambda_i P_i \equiv \left\{ \mu \in \triangle S \;\middle|\; \exists \mu_i \in P_i \text{ for } i = 1, \ldots, N: \; \mu = \sum_{i=1}^{N} \lambda_i \mu_i \right\} \tag{4}$$

Finally, we formalize the idea of combining different aggregation approaches.

DEFINITION 8. *Let $\mathscr{A} \subseteq \Pi$ be a non-empty closed collection of frames. Then its closure $\Gamma(\mathscr{A})$ with respect to the operations of convex union, convex combination and coherent intersection is a minimum family of frames that satisfies (i) $\mathscr{A} \subseteq \Gamma(\mathscr{A})$, and (ii) if P is either a convex union, or a convex combination, or a coherent intersection of a collection of frames $\mathscr{C} \subseteq \Gamma(\mathscr{A})$, then $P \in \Gamma(\mathscr{A})$.*

Lemma 20 in Appendix shows that the closure given by Definition 8 exists and is unique. Our third main result is that, when the number of frames is finite, the three considered operations of frame connection characterise the comparative consistency:

Theorem 3. *Let $c_1(\cdot)$ and $c_2(\cdot)$ be represented by $(U_1, \mathscr{A}_1)$ and $(U_2, \mathscr{A}_2)$ respectively, and $|\mathscr{A}_2| < \infty$. Then the following statements are equivalent:*

(i) DM 1 is more consistent than DM 2;

(ii) U_1 is a positive affine transformation of U_2, and $\mathscr{A}_1 \subseteq \Gamma(\mathscr{A}_2)$;

(iii) U_1 is a positive affine transformation of U_2, and any $P \in \mathscr{A}_1$ is a coherent intersection of convex unions of convex combinations of frames in $\mathscr{A}_2$.

If $\mathscr{A}_2$ is infinite, we might need to generalize the definition of the convex combination of frames to work with infinite closed collections of frames[20]. We conjecture that Theorem 3 remains true with infinite number of frames as well.

[20] Weights λ generalize to the probability distribution on the appropriately defined sigma-algebra of collections of frames, and selection of priors $\mu_i \in P_i$ generalizes to the measurable selection of priors from the collection.

4.3. Identification of Frames from Preference Relation

Ambiguity literature commonly considers the primitive to be a preference relation on the set of Anscombe-Aumann acts, and not a choice correspondence, as we do. In this section, we analyze to which extend the analyst can identify the frames from the observed preference relation.

We begin our analysis by showing that data on choices from binary menus is sufficient to conclude that one agent is more consistent than another according to Definition 8.

Proposition 4. *Let $c_1(\cdot)$ and $c_2(\cdot)$ satisfy axioms 1–7. Then DM 1 is more consistent than DM 2 if and only if $\forall g,h \in H$ $c_1(\{f,g\}) \subseteq c_2(\{f,g\})$.*

Thus, DM 1 is more consistent than DM 2 if and only if DM 1 is more decisive than DM 2 on the set of binary menus. This binary formulation of comparative consistency allows us to use Theorem 3 to get identification result for the representation of a preference relation.

Formally, model $(U,\mathscr{A})$ represents preference relation $\succeq$ if $(U,\mathscr{A})$ represents choice correspondence $c(\cdot)$ such that $f \succeq g \iff f \in c(\{f,g\})$ $\forall f,g \in H$. Note that $c_1(\cdot) = c_2(\cdot)$ for binary menus if and only if DM 1 is more consistent than DM 2, and DM 2 is more consistent than DM 1; hence:

Corollary 2. *Framed ambiguity models $(U_1,\mathscr{A}_1)$ and $(U_2,\mathscr{A}_2)$ with $|\mathscr{A}_1|,|\mathscr{A}_2| < \infty$ represent the same preference relation if and only if U_2 is a positive affine transformation of U_1 and $\Gamma(\mathscr{A}_2) = \Gamma(\mathscr{A}_1)$.*

Corollary 2 tells us that observing choices only from binary menus leaves a big room for different representations. Moreover, in contrast to Theorem 2 for choice correspondence, an analyst may not be able to identify a unique minimum collection of frames that represents[21] a given preference relation, which is clear from the following example, illustrated in Figure 1.

Let $X = \{x,y\}$, $U(p) = p(x)$, and $|S| = 3$. Consider priors $\mu_1 = (0.1,0.2,0.7)$, $\mu_2 = (0.3,0.2,0.5)$, $\mu_3 = (0.2,0.3,0.5)$, $\mu_4 = (0.2,0.1,0.7)$ and sets of priors $P_1 = \{\mu_1\}$, $P_2 = \{\mu_2\}$, $P_3 = conv(\{\mu_3,\mu_4\})$, $P_5 = conv(\{\mu_1,\mu_3,\mu_4\})$, $P_6 = conv(\{\mu_2,\mu_3,\mu_4\})$. Let the first family of frames be $\mathscr{A}_1 = \{P_1,P_2,P_3\}$, and the second family be $\mathscr{A}_2 = \{P_1,P_2,P_5,P_6\}$. Note that $P_5 = conv(P_1 \cup P_3)$, $P_6 = conv(P_2 \cup P_3)$, and $P_3 = P_5 \cap P_6$, where the intersection of P_5 and P_6 is coherent (Lemma 29). Hence, $\Gamma(\mathscr{A}_1) = \Gamma(\mathscr{A}_2)$. By Corollary 2, models $(U,\mathscr{A}_1)$ and $(U,\mathscr{A}_2)$ represent the same preference relation $\succeq$.

Preference relation $\succeq$ does not have a representation with minimum subset of frames, since in all candidate models $(U,\{P_1,P_2\}),(U,\{P_1\}),(U,\{P_2\})$, DM's belief regarding $s = 2$ is fixed: $\mu(2) =$

[21] Nothing changes for the identification of the vNM expected utility function, since the restriction $c_0(\cdot)$ of choice correspondence on the set $\mathscr{K}_0$ of menus consisting of constant acts satisfies *WARP*.

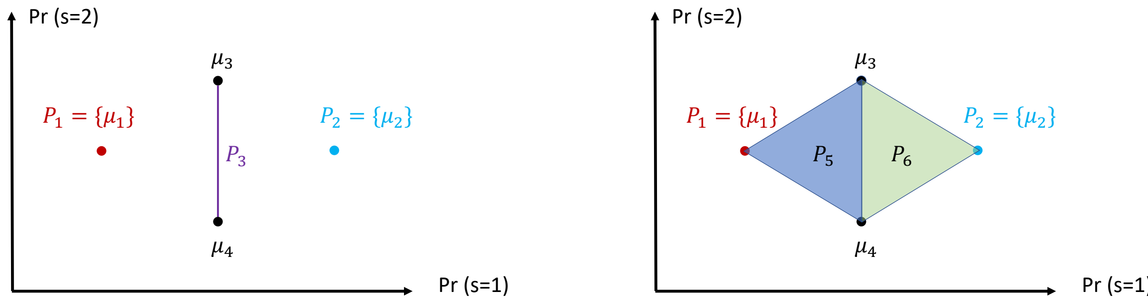

Figure 1

$\mu_1(2) = \mu_2(2) = 0.2$, while the original models allow DM to use priors $\mu_3(2) = 0.3$ and $\mu_4(2) = 0.1$. Hence, the candidate models cannot represent $\succeq$.

By Theorem 2, $(U, \mathscr{A}_1)$ and $(U, \mathscr{A}_2)$ represent different choice correspondences $c_1(\cdot)$ and $c_2(\cdot)$. For example, let $f, g \in H$, $p \in H_0$ be as follows: $f(x) = (1,0,0)$, $g(x) = (0,1,0)$, $p(x) = 0.15$, then $c_1(\{f,g,p\}) = \{f,g\}$, but $c_2(\{f,g,p\}) = \{f,g,p\}$, where p is chosen under frame P_5 for c_2. By Theorem 2, $\mathscr{A}_1$ is the unique family of frames representing $c_1(\cdot)$; $c_2(\cdot)$ admits two different representations: $\mathscr{A}_2$ is the minimum family of frames, and $(\mathscr{A}_2)_{coh} = \mathscr{A}_2 \cup \{P_3\}$ is the maximum family of frames that represents $c_2(\cdot)$.

4.4. Application to Aggregation of Preferences

Our model has a population interpretation: consider a group of people that agree on the ranking of prizes (share the same utility), are averse to ambiguity, and disagree on beliefs. Suppose people in this group want to aggregate their judgements such that the resulting rule is rational, respects the ambiguity aversion, and a natural *Unanimity* criterion:

DEFINITION 9. *Let $\{\succeq_i\}_{i=1,\dots,N}$ and $\succeq$ be binary relations on H. We say that $\succeq$ satisfies Unanimity with respect to $\{\succeq_i\}_{i=1,\dots,N}$ if for all $f, g \in H$ $[f \succeq_i g \ \forall i = 1,\dots,N] \implies f \succeq g$.*

Theorem 3 helps us to characterize such rules:

Corollary 3. *Let $\{\succeq_i\}_{i=1,\dots,N}$ and $\succeq$ be preference relations on H that have Gilboa and Schmeidler (1989) maxmin representations with the same vNM expected utility function, but different sets of beliefs $\{P_i\}_{i=1,\dots,N}$ and P. Then $\succeq$ satisfies Unanimity with respect to $\{\succeq_i\}_{i=1,\dots,N}$ if and only if P is a coherent intersection of convex unions of convex combinations of $\{P_1,\dots,P_N\}$.*

Note also that, according to Theorem 3, taking arbitrary number of operations of coherent intersection, convex union and convex combination of $\{P_1, ..., P_N\}$ in any order always produces a set of beliefs that corresponds to a rule in the desired class.

Aggregation of ambiguity averse opinions in the literature. In Crès et al. (2011), the authors study the aggregation of the opinions of a group of ambiguity-averse experts. They impose a stronger *Expert Uncertainty Aversion* axiom. In terms of our result, their aggregation procedure is equivalent to taking convex unions and convex combinations of sets of beliefs, but not coherent intersections. We show that usage of all three of these operations characterises the aggregation procedure that satisfies the Unanimity criterion. To achieve this result, we use some of the instruments from the proofs in Crès et al. (2011) and Chandrasekher et al. (2022).

In Hill (2011), the author characterizes the same aggregation rule as Crès et al. (2011) in terms of the Unanimity criterion plus the requirement that an aggregation rule should remain the same across all profiles of preferences. In terms of our model, Hill (2011)'s additional requirement rules out coherent intersections primarily because not all intersections of sets of beliefs are coherent, hence if a rule includes a coherent intersection, it cannot be applied to all profiles of preferences.

4.5. Optimistic Learning

Consider an agent who contemplates her potential decisions under the various decision frames. Let she face, for example, a choice between an act $f = \sigma f_1 + (1-\sigma) f_2$ and a constant act x (money). She figures that under frame 1, she would prefer f_1 to x to f_2, while under frame 2, f_2 to x to f_1. Thinking optimistically, she picks frame 1 to evaluate f_1, frame 2 to evaluate f_2 and concludes that act f is better than x.

Our next proposition bounds the beliefs of the rational ambiguity-averse agent (DM 1) who applies such "optimistic" arguments either to experts' suggestions—with the interpretation that each frame is an expert's advice—or to her potential frame-susceptible choice behavior (DM-2). For brevity, we omit the universal qualifiers $\forall \{f_i\} \subset H, \forall p \in H_0$ in statement (i) below.

Proposition 5. *Let $c_2(\cdot)$ be represented by the framed ambiguity model $(U, \mathscr{A})$, and $c_1(\cdot)$ be represented by the Gilboa and Schmeidler (1989) maxmin model with vNM expected utility V and set of beliefs P. Then the following statements are equivalent:*

(i) If $f_i \in c_2(\{f_i, p\})$ for all $i = 1, .., k$, then $\sum_i^k \sigma_i f_i \in c_1\left(\left\{\sum_i^k \sigma_i f_i, p\right\}\right)$ for all convex weights σ.

(ii) V is a positive affine transformation of U, and $P \subseteq \bigcap_{Q \in \mathscr{A}} Q \neq \varnothing$.

Proposition 5 says that an ambiguity-averse decision maker who contemplates a set of frames $\mathscr{A}$ and finds "arguments" $f_i \in c_2(\{f_i, p\})$ to evaluate a mixture $\sum_i^k \sigma_i f_i$ to be at least as good as act p should consider only priors that lie in the intersection of all frames in $\mathscr{A}$—but not necessary all these priors. If the intersection of frames in $\mathscr{A}$ is empty, this behavior becomes inconsistent with minimization of expected utility over a set of priors—the agent becomes too optimistic.

Finally, we analyze the case when the reverse implication in statement (i) of Proposition 5 holds as well—with the nuance that the agent is also capable of applying the *C-Independence* principle to guide her decisions. Thus, for instance, the agent chooses act f in comparison to a constant act p only if she finds a decomposition of the form $\lambda f + (1-\lambda)q = \sum_{i=1}^k \sigma_i f_i$, where q is some constant act[22] such that each f_i can be chosen in comparison to $\lambda p + (1-\lambda)q$ under some frame. We call such way of reasoning *optimistic learning* and characterize it in our final result; for this result to hold, it suffices to require one of the two following conditions[23].

CONDITION 1. *A family of frames $\mathscr{A}$ is finite, $\bigcap_{P \in \mathscr{A}} P \neq \varnothing$, and each $P \in \mathscr{A}$ is polyhedral.*

Condition 1 says, in particular, that each $P \in \mathscr{A}$ can be expressed via a finite number of linear inequalities. Equivalently, each $P \in \mathscr{A}$ is a polytope—that is, there are finitely many priors $\mu \in P$ that DM uses to evaluate acts under each frame P.

CONDITION 2. *A family of frames $\mathscr{A}$ is finite, and $\bigcap_{P \in \mathscr{A}} ri(P) \neq \varnothing$.*

Here, we denote by $ri(P)$ the relative interior of the set P. Intuitively, Condition 2 says that the intersection of frames does not cause the ambiguity existing in each frame in some dimension to vanish. The following proposition uses results regarding the dual of a sum of convex functions[24]. To ease notations, we omit the universal quantifiers $\forall f \in H, \forall p \in H_0$ in statement (i); it is assumed that p and q are constant acts, f_i are general acts, and σ_i are convex weights.

Proposition 6. *Let $c_2(\cdot)$ be represented by the framed ambiguity model $(U, \mathscr{A})$, and either Condition 1 or Condition 2 holds for $\mathscr{A}$. Then the following statements are equivalent:*

(i) $c_1(\cdot)$ satisfies WARP and Continuity, and $f \in c_1(\{f, p\})$ if and only if there exists a decomposition

[22] The reason why we consider a mixture of f with a constant act q is that in our setup, the set of acts $H = (\triangle X)^S$ have extreme points that do not have non-trivial decompositions. If the set of consequences is instead $\mathbb{R}^N$, and the agent's utility function is linear over consequences, considering this extra mixture is not necessary.

[23] If we instead demand just that $\mathscr{A}_2$ is finite (which would not matter) and has a non-zero intersection, then a generalized version of the proposition would require consideration of limits of choices made by DM 2.

[24] Corollary 16.4.1 and Theorem 20.1 in Rockafellar (1970).

$\lambda f + (1-\lambda)q = \sum_{i=1}^{k} \sigma_i f_i$ with $\lambda \in (0,1]$ such that $f_i \in c_2(\{f_i, \lambda p + (1-\lambda)q\})$ for all $i = 1, \ldots, k$;

(ii) $c_1(\cdot)$ is represented by the Gilboa and Schmeidler (1989) maxmin model with vNM expected utility function U and set of priors $\bigcap_{Q \in \mathscr{A}} Q$.

Proposition 6 says that, under a mild technical condition, the agent who perceives the relevant ambiguity to be the intersection of frames in $\mathscr{A}$ is the agent who evaluates an act using its most favorable decomposition according to the variety of decision frames $\mathscr{A}$.

5. Conclusion and Discussion

Experimental literature suggests that, first, framing of uncertainty can matter for the agents' choice behavior and revealed attitude towards ambiguity. Schneider et al. (2018) and Leland et al. (2019) find that the degree of ambiguity aversion depends on the transparency of representation of payoffs in Ellsberg urn problems. Maher and Kashima (1997) and Esponda and Vespa (2016) find that representing an Ellsberg urn problem in contingent versus non-contingent manner matters for agents' decisions[25].

Second, framing of the relevant state space can matter for the formation of beliefs. A classical study by Fischhoff et al. (1978) finds that even experienced mechanics assign different probabilities for the same cause—for instance, "Fuel system"—that leads to the event "Car does not start," depending on the structure of the "Fault Tree" representing these potential reasons of car not starting. Fox and Rottenstreich (2003) and Fox and Clemen (2005), for environments including uncertainty in weather, sports, and business, also find that beliefs assigned to events differ depending on how the state space is partitioned. Sonnemann et al. (2013) finds that this dependence on partition of the state space matters for the market prices as well.

Instead of modeling framing explicitly as, for instance, in Ahn and Ergin (2010), our paper considers framing to be unobservable, as is often the case in the real world. We build a model that incorporates both the ideas that framing of uncertainty matters for belief formation, but also for choice under ambiguity. To obtain this, we extend Gilboa et al. (2010) maxmin model by allowing the set of beliefs to depend on the framing that is unobservable to the analyst.

Our main identification result is surprisingly sharp: there exists a unique minimum collection of decision frames, operationalized via sets of beliefs, that is consistent with the observable data.

[25] Martínez-Marquina et al. (2019) find that contingent/non-contingent framing matters in more simple environments as well, attributing this effect to the "Power of Certainty."

Crucially, as discussed in Section 4.3, the analyst needs to leverage the data on choices from larger than binary menus to obtain this identification. This is not the case, for instance, for models that assume *Normality* of the choice[26]. In such models, all relevant information about the parameters is contained in the preference relation (choices from binary menus).

To prove our characterization and identification results (Theorems 1,2), we developed a novel—to the best of our knowledge—technique that can be applied to identify unobservable decision frames. At the first step, the analyst can find, for each alternative, the *collection* of menus that are *maximal* with respect to the property that the given alternative can be chosen from the menu. This concept generalizes the lower contour set of the alternative for a rational decision maker satisfying *WARP*, in which case this collection consists of a single menu—the lower contour set of the given alternative. At the second step, we show that—in our setup—there is a surjective mapping from the collection of maximal menus across alternatives onto the minimum collection of the decision frames representing the decision maker's choice.

Note that even if some framing in the experiment is controlled, it might be hard to exclude all other types of framing, that is, details of the environment that influence the agent's decisions under ambiguity. Our analysis suggests to use non-binary menus in the experiments, since choices from larger menus contain additional information that cannot be inferred from the binary comparisons if framing of uncertainty is involved.

Appendix

Proof of Theorem 1

For brevity, we omit the proof that axioms follow from the representation except of the proof for the *Indirect Ambiguity Aversion* and *Continuity* axioms. Assume $(U,\mathscr{A})$ represents $c(\cdot)$.

Let $h \in A$, $h \notin c(A \cup \{f\})$, $h \notin c(A \cup \{g\})$, $\mathscr{C} = \{P \in \mathscr{A} | W_P(h) \geq W_P(h') \forall h' \in A\}$. Then $W_P(h) < W_P(f), W_P(g)\ \forall P \in \mathscr{C}$, hence $W_P(\lambda f + (1-\lambda)g) \geq \lambda W_P(f) + (1-\lambda)W_P(g) > W_P(h)\ \forall P \in \mathscr{C}$. Therefore, $\nexists P \in \mathscr{A} : W_P(h) \geq W_P(h')\ \forall h' \in A \cup \{\lambda f + (1-\lambda)g\}$, $h \notin c(A \cup \{\lambda f + (1-\lambda)g\})$. Therefore, *Indirect Ambiguity Aversion* holds. □

Let $(A^k, f^k) \longrightarrow (A, f) : f^k \in c(A^k)\ \forall k = 1,2,\ldots$ Then $f^k \in A^k$, and $\rho(\{f\}, A) \leq d(f, f^k) + \rho(\{f^k\}, A) \leq d(f, f^k) + \rho(A^k, A) \longrightarrow 0$, hence $f \in cl(A) = A$. Since $f^k \in c(A^k)$ for $k = 1,2,\ldots$, $\exists P^k \in \mathscr{A}$:

[26] If we impose *Normality* for our model, it reduces to Gilboa et al. (2010) maxmin model; in other words, framing becomes trivial (Proposition 2).

$W_{P^k}(f^k) \geq W_{P^k}(h)$ $\forall h \in A^k$. Since Π is compact, $\exists P^{k_m} \longrightarrow P \in \Pi$. Since $\mathscr{A}$ is closed, $P \in \mathscr{A}$. Consider arbitrary $g \in A$; since $A^{k_m} \longrightarrow A$, $\exists g^{k_m} \in A^{k_m}$: $g^{k_m} \longrightarrow g$. Then $W_{P^{k_m}}(f^{k_m}) \geq W_{P^{k_m}}(g^{k_m})$ $\forall m$. Note that the function $\zeta : \Pi \times H \times H$ given by

$$\zeta(Q,h,h') = \min_{\mu \in P} \sum_{s \in S} \mu_s U(f(s)) - \min_{\mu \in P} \sum_{s \in S} \mu_s U(g(s))$$

is continuous, hence $W_P(f) \geq W_P(g)$, and $f \in c(A)$. Therefore, *Continuity* holds. □

Assume now that choice correspondence $c(\cdot)$ satisfies Axioms 1-7. We show that $c(\cdot)$ admits a framed ambiguity representation. The next lemma is strightforward.

Lemma 1. *Continuity (Axiom 6) implies (i)* ***Act Continuity:*** $\forall A \in \mathscr{K}$ $\{f \in H \mid f \in c(A \cup \{f\})\}$ *is closed, (ii)* ***Menu Continuity:*** $\forall f \in H$ $\{A \in \mathscr{K} \mid f \in c(A)\}$ *is closed.*

We say that the restriction of choice correspondence c_0 on the set of compact menus of constant acts $\mathscr{K}_0$ has an expected utility representation, if there is a vNM expected utility $U : \triangle X \to R$ such that $\forall A \in \mathscr{K}_0$ $[(p,...,p) \in c(A) \iff [U(p) \geq U(q) \;\; \forall q \in A]]$. Note that if $c(\cdot)$ satisfies Axioms 1-7, so does $c_0(\cdot)$.

Lemma 2. *Given Axioms 1-7, c_0 has a non-degenerate expected utility representation U that is unique up to a positive affine transformation. Moreover, $f(s) \in c(f(s), g(s)) \Longleftrightarrow U(f(s)) \geq U(g(s))$.*

Proof. Consider binary relation $\succeq_{c_0} \subseteq H_0 \times H_0$ given by $p \succeq_{c_0} q$ if $p \in c_0(\{p,q\})$. Axioms α and *C-β* imply *WARP* for c_0, thus $\succeq_{c_0}$ is complete and transitive. The *C-Independence* for $c_0(\cdot)$ implies Independence for $\succeq_{c_0}$. *Act Continuity* implies $\{q \in H \mid p \succeq_{c_0} q\}$ is closed, and *Menu Continuity* implies $\{q \in H \mid p \preceq_{c_0} q\}$ is closed $\forall p \in H_0$; these statements imply the Archimedean continuity of $\succeq_0$. The expected utility representation follows from the Mixture Space Theorem. Since *C-Non-Degeneracy* implies $p \not\succeq_0 q$ for some p, q, this representation is non-degenerate. The last assertion follows from the definition of U. □

Lemma 2 allows us to find prizes x^* and x_* such that $u^* \equiv U(x^*) > U(x_*) \equiv u_*$, where we abused notations by identifying prize x with lottery δ_x. Given menu B, denote by

$$\overline{x}(B) = \operatorname*{arg\,max}_{x \in \left\{x \middle| \exists f \in B, s \in S:\; f(s)(x) > 0\right\} \cup \{x^*\}} U(x), \quad \underline{x}(B) = \operatorname*{arg\,min}_{x \in \left\{x \middle| \exists f \in B, s \in S:\; f(s)(x) > 0\right\} \cup \{x_*\}} U(x)$$

and $\overline{u}(B) = U(\overline{x}(B))$, $\underline{u}(B) = U(\underline{x}(B))$. Note that the set $\left\{x \middle| \exists f \in B, s \in S :\; f(s)(x) > 0\right\}$ is finite by the definition of the menu, thus the introduced above quantities are well-defined. We will also sometimes omit the dependence of $\overline{x}$, $\underline{x}$, $\overline{u}$, $\underline{u}$ on B when it will not cause confusion.

Lemma 3. *If $c(\cdot)$ satisfies Axioms 1-7, then it satisfies* ***Monotonicity:*** $\forall f,g \in H$, $\forall A \in \mathcal{K}$ *if* $f(s) \in c(\{f(s),g(s)\})$ $\forall s \in S$ *then (i)* $g \in c(A) \implies f \in c(A \cup \{f\})$*; (ii)* $f \in A \implies c(A) \subseteq c(A \cup \{g\})$.

Proof. Let $f(s) \in c(\{f(s),g(s)\})$ $\forall s \in S$. Then by Lemma 2, $U(f(s)) \geq U(g(s))$ $\forall s \in S$. Given menus A, B, and act f, denote by

$$A_{0.5} \equiv 0.5A + 0.5 \cdot (0.5\overline{x}(B) + 0.5\underline{x}(B)), \qquad f_{0.5} \equiv 0.5f + 0.5 \cdot (0.5\overline{x}(B) + 0.5\underline{x}(B))$$

Proof of (i). Consider $B = A \cup \{f\}$ and $g \in c(A)$, then by *C-Independence*, $g_{0.5} \in c(A_{0.5})$. Consider $f^n_{0.5} \equiv (1-1/n)f_{0.5} + (1/n)\overline{x}$ for $n = 1,2,\ldots$ Then $\forall s \in S$ we have:

$$U(f^n_{0.5}(s)) = \left(1-\frac{1}{n}\right) \cdot \left(\frac{U(f(s))}{2} + \frac{\overline{u}+\underline{u}}{4}\right) + \frac{\overline{u}}{n} \geq \frac{U(f(s))}{2} + \frac{\overline{u}-\underline{u}}{4n} > U(f_{0.5}(s)) \geq U(g_{0.5}(s))$$

By Lemma 2, $g_{0.5}(s) \notin c(\{g_{0.5}(s), f^n_{0.5}(s)\})$ $\forall s \in S$. *Strict Monotonicity* implies $g_{0.5} \notin c(\{g_{0.5}, f^n_{0.5}\})$, then by α, we have $g_{0.5} \notin c(A_{0.5} \cup \{f^n_{0.5}\})$. By *Aizerman's Property*, $f^n_{0.5} \in c(\{A_{0.5} \cup f^n_{0.5}\})$. Since $f^n_{0.5} \longrightarrow f_{0.5}$, by *Act Continuity*, $f_{0.5} \in c(A_{0.5} \cup \{f_{0.5}\})$. By *C-Independence*, $f \in c(A \cup \{f\})$.

Proof of (ii). Consider $B = A \cup \{g\}$, $f \in A$, and $h \in c(A)$. By *C-Independence*, $h_{0.5} \in c(A_{0.5})$. Consider $g^n_{0.5} \equiv (1-1/n)f_{0.5} + (1/n)\underline{x}$ for $n = 1,2,\ldots.$ Then $\forall s \in S$ we have:

$$U(g^n_{0.5}(s)) = \left(1-\frac{1}{n}\right) \cdot \left(\frac{U(g(s))}{2} + \frac{\overline{u}+\underline{u}}{4}\right) + \frac{\underline{u}}{n} \leq \frac{U(g(s))}{2} - \frac{\overline{u}-\underline{u}}{4n} < U(g_{0.5}(s)) \leq U(f_{0.5}(s))$$

By Lemma 2, $g^n_{0.5}(s) \notin c(\{g^n_{0.5}(s), f_{0.5}(s)\})$ for all $s \in S$, by *Strict Monotonicity*, $g^n_{0.5} \notin c(\{g^n_{0.5}, f_{0.5}\})$, by α, we have $g^n_{0.5} \notin c(A_{0.5} \cup \{g^n_{0.5}\})$. By *Aizerman's Property*, $h_{0.5} \in c(A_{0.5} \cup \{g^n_{0.5}\})$. Next, $A_{0.5} \cup \{g^n_{0.5}\} \longrightarrow A_{0.5} \cup \{g_{0.5}\}$, hence by *Menu Continuity*, $h_{0.5} \in c(A_{0.5} \cup \{g_{0.5}\})$. Finally, by *C-Independence*, $h \in c(A \cup \{g\})$. $\square$

Let us show that it is without loss of generality to focus on acts with a binary set of prizes $\{x^*, x_*\}$.

Lemma 4. *If Axioms 1-7 hold for $c(\cdot)$, then $\forall A \in \mathcal{K}$ $\forall f \in A$, $f \in c(A) \iff T_A(f) \in c(T_A(A))$, where* $T_A(f) = (\alpha_s x^* + (1-\alpha_s)x_*)_{s \in S}$, $\alpha_s = \dfrac{\lambda(A) \cdot U(f(s)) + (1-\lambda(A)) \cdot (0.5u^* + 0.5u_*) - u_*}{u^* - u_*}$, $\lambda(A) = 0.1 \cdot \min\left\{1, \dfrac{u^* - u_*}{|\overline{u}(A) - 0.5u^* - 0.5u_*|}, \dfrac{u^* - u_*}{|\underline{u}(A) - 0.5u^* - 0.5u_*|}\right\}$, *and* $T_A(B) = \{T_A(g) | g \in B\}$.

Proof. First, note that $\lambda \in [0, 0.1]$, and $\alpha_s \in [0.4, 0.6]$, thus $T_A(f)$ is a well-defined act, and $T_A(A)$ is a set of acts. Since $U(\cdot)$ is continuous, so does T_A; hence, since A is compact, $T_\lambda(A)$ is compact as well. Note also that acts in $T_A(A)$ yield only two prizes. Therefore, $T_A(A)$ is a menu.

Denote by $f \succeq^* g$ if $U(f(s)) \geq U(g(s))$, and $f \sim g$ if $[f \succeq^* g$ and $g \succeq^* f]$.

Claim 1. *Let $g \in B$, $h \sim g$. If $g \in c(B)$, $c(B \cup \{h\}) = c(B) \cup \{h\}$; if $g \notin c(B)$, then $c(B \cup \{h\}) = c(B)$.*

Proof of Claim 1. Let $g \in c(B)$. If $h \notin c(B \cup \{h\})$, by *Aizerman's Property*, $g \in c(B \cup \{h\})$, and by *Monotonicity*, $h \in c(B \cup \{h\})$. Since $g \succeq^* h$, by *Monotonicity*, $c(B) \subseteq c(B \cup \{h\})$. By α, $c(B \cup \{h\}) = c(B) \cup \{h\}$. Let $g \notin c(B)$. If $h \in c(B \cup \{h\})$, then by *Monotonicity*, $g \in c(B \cup \{h\})$, and by α, $g \in c(B)$, contradiction. Hence, $h \notin c(B \cup \{h\})$, and by *Aizerman's Property* and α, $c(B \cup \{h\}) = c(B)$. ☐

Claim 2. *Let $\{h_1, \ldots, h_k\} \subset H$. Assume $\forall i \in \{1, \ldots, k\}\ \exists g_i \in A$: $g_i \sim h_i$. Then:*

$$c(A \cup \{h_1, \ldots, h_k\}) \;=\; c(A) \cup \big\{h \in \{h_1, \ldots, h_k\} \;\big|\; \exists g \in c(A) : h \sim g\big\}$$

Proof of Claim 2. Let us use the induction in $k = 0, 1, \ldots$ When $k = 0$, the statement $c(A) = c(A)$ is true. For the induction step, if $h_{k+1} \sim g$ for $g \in A$, then the statement follows from Claim 1 and the induction hypothesis. If $h_{k+1} \sim h_i$ for $i \in \{1, \ldots, k\}$, then by the induction hypothesis and transitivity of $\sim$, there is $g_i \in A$ such that $h_{k+1} \sim g_i$ and the same argument applies. ☐

Since A is compact, $\exists F_n$: $F_n \subseteq A \subseteq \bigcup\limits_{f \in F_n} \{g \in H |\ d(g, f) < 1/n\}$. Denote by $F^n \equiv \bigcup\limits_{m=1}^{n} F_m$.

Claim 3. *(i) $F^n \cup T_A(A) \longrightarrow A \cup T_A(A)$, (ii) $A \cup T_A(F^n) \longrightarrow A \cup T_A(A)$.*

Proof of Claim 3. Statement (i) is straightforward. Next, note that for any $s \in S$, the mapping $f \to U(f(s))$ is uniformly continuous on A, since $U(\cdot)$ is linear, $\underline{u}(A) \le U(f(s)) \le \overline{u}(A)$, and A is compact. It follows that $f \to T_A(f)$ is uniformly continuous on A, implying

$$\rho(A \cup T_A(F^n), A \cup T_A(A)) \;=\; \max_{h \in A \cup T_A(A)} \; \min_{g \in A \cup T_A(F^n)} d(g,h) \le \max_{h \in T_A(A)} \; \min_{g \in T_A(F^n)} d(g,h) \;\longrightarrow\; 0 \quad ☐$$

Denote by $g_\lambda = \lambda(A) g + (1 - \lambda(A))(0.5x^* + 0.5x_*)$ and $A_\lambda = \lambda(A)A + (1 - \lambda(A))(0.5x^* + 0.5x_*)$. By construction, $U(g_\lambda(s)) = U(T_A(g)(s))$ for all $s \in S$, hence $g_\lambda \sim T_A(g)$ for all $g \in A$.

Assume $f \in c(A)$. By *C-Independence*, $f_\lambda \in c(A_\lambda)$. WLOG, $f \in F_1 \subseteq F^n$. By Claim 2, $f_\lambda \in c(A_\lambda \cup T_A(F^n))$. By Claim 3 and *Menu Continuity*, $f_\lambda \in c(A_\lambda \cup T_A(A))$. By *Monotonicity*, $T_A(f) \in c(A_\lambda \cup T_A(A))$. By α, $T_A(f) \in c(T_A(A))$.

Assume $T_A(f) \in c(T_A(A))$. WLOG, $f \in F_1 \subseteq F^n$. By Claim 2, $T_A(f) \in c(T_A(A) \cup F^n_\lambda)$. By Claim 3 and *Menu Continuity*, $T_A(f) \in c(T_A(A) \cup A_\lambda)$. By *Monotonicity*, $f_\lambda \in c(T_A(A) \cup A_\lambda)$. By α, $f_\lambda \in c(A_\lambda)$. By *C-Independence*, $f \in c(A)$, proving Lemma 4. ☐

We say that a framed ambiguity model $(U, \mathscr{A})$ represents $c(\cdot)$ on the set of menus $\overline{\mathscr{K}}$ if for any menu $B \in \overline{\mathscr{K}}$, $c(B)$ is given by eq. (1), (2).

Lemma 5. *A framed ambiguity model $(U, \mathscr{A})$ represents $c(\cdot)$ on $\mathscr{K}$ if and only if $(U, \mathscr{A})$ represents $c(\cdot)$ on the set of menus $\mathscr{K}^*$ consisting of acts that yield only prizes x^*, x_*.*

Proof. One direction is trivial. Assume now that $(U,\mathscr{A})$ represents $c(\cdot)$ on $\mathscr{K}^*$. Then Axioms 1-7 hold for the choice correspondence $\hat{c}$ induced by $(U,\mathscr{A})$ according to eq. (1), (2). Note that $\hat{c}(B)=c(B)$ for all $B\in\mathscr{K}^*$. Therefore, by Lemma 4, $f\in\hat{c}(A)\Longleftrightarrow T_A(f)\in\hat{c}(T_A(A))\Longleftrightarrow T_A(f)\in c(T_A(A))\Longleftrightarrow f\in c(A)$. Hence, $\hat{c}=c$, proving the other direction. □

By Lemma 5, WLOG, $X=\{x^*,x_*\}$, $U(x^*)=1$, $U(x_*)=0$. In this case, the set of acts H is isomorphic to the hypercube $[0,1]^S$. We will use notations $\mathbf{z},f$ for general acts and $H=[0,1]^S$. Denote by $\iota=(1,...,1)\in R^S$, then act $\mathbf{z}\in[0,1]^S$ is constant if and only if $\mathbf{z}=\overline{z}\iota$ for some $\overline{z}\in[0,1]$.

Lemma 6. *Let $X=\{x^*,x_*\}$ and let Menu Continuity hold. Then $\forall f\in H,\forall A\in\mathscr{K}$ if $f\in c(A)$, then $\exists D\in\mathscr{K}$: $\big[A\subseteq D$, $f\in c(D)$, and if $[D'\in\mathscr{K}, D'\neq D$, and $D\subseteq D']$, then $f\notin c(D')\big]$.*

Proof. Consider the set of menus $\mathscr{Q}(A,f)\equiv\{B\in\mathscr{K}\mid f\in c(B)\text{ and }A\subseteq B\}$ partially ordered by set inclusion. Let $\mathscr{T}\subseteq\mathscr{Q}(A,f)$ be an arbitrary non-empty totally ordered set. Let $C=cl\,(\bigcup_{B\in\mathscr{T}}B)$. Note that C is compact since it is a closed subset of the compact set H. Therefore, for any $\epsilon>0$, there is a finite set $\{f_i\}_{i\in\{1,...,N\}}\subseteq C$ such that $\forall h\in C$ $d(f_i,h)<\epsilon/3$ for some $f_i=f_i(h)$. Next, by the definition of C, for any $i\in\{1,...,N\}$, there is $B_i\in\mathscr{C}$ and $g_i\in B_i$ such that $d(g_i,f_i)<\epsilon/3$. Since $\mathscr{T}$ is totally ordered by set inclusion, there is $\widetilde{B}\in\{B_1,...,B_N\}$ such that $B_i\subseteq\widetilde{B}$ for all $i=\{1,...,N\}$. Since $\widetilde{B}\subseteq C$, it follows

$$\rho(\widetilde{B},C)\;=\;\sup_{h\in C}\left(\inf_{g\in\widetilde{B}}d(g,h)\right)\;\leq\;\sup_{h\in C}\big(d(g_i,f_i(h))+d(f_i(h),h)\big)\;\leq\;\epsilon/3+\epsilon/3<\epsilon$$

Consider sequence $\epsilon\longrightarrow 0$, and the associated menus $\widetilde{B}_\epsilon\in\mathscr{T}$ with $\rho(\widetilde{B}_\epsilon,C)<\epsilon$. Since $f\in c(\widetilde{B}_\epsilon)$, $\widetilde{B}_\epsilon\longrightarrow C$, and $C\in\mathscr{K}$, by *Menu Continuity*, $f\in c(A)$. Thus, every non-empty totally ordered subset $\mathscr{T}$ of $\mathscr{Q}(A,f)$ has an upper bound $C\in\mathscr{Q}(A,f)$. Note also that $A\in\mathscr{Q}(A,f)\neq\varnothing$. By Zorn's Lemma, $\mathscr{Q}(A,f)$ has at least one maximal element D. □

Given $\lambda\in[0,1]$, let $f_\lambda=\lambda f+(1-\lambda)0.5\iota$, $B_\lambda=\lambda B+(1-\lambda)0.5\iota$. For $f\in H_{0.2}$, let $\mathscr{L}(f)\equiv\big\{A\in\mathscr{K}\,|\,f\in c(A)\text{ and }[A\subseteq A'\text{ and }f\in c(A')\implies A'=A]\big\}$. Since $f\in c(\{f\})$, by Lemma 6, $\mathscr{L}(f)\neq\varnothing$. For $A\in\mathscr{L}(f)$, denote by

$$a(A)=\max\{b\in[0,1]\mid b\cdot\iota\in A\}\tag{5}$$

Since A is compact and $0\iota\in A$, by *Monotonicity*, $a(A)$ is well-defined; moreover, $0.4\leq a(A)\leq$ since $f\in H_{0.2}$. Denote by $|\mathbf{z}|=\max_{s\in S}|z_s|$, for $\mathbf{v}\in\mathbb{R}^S$: $\sum_{s\in S}v_s=0$ and $|\mathbf{v}|=1$, define

$$J_A(\mathbf{v})=-\max\{b\in\mathbb{R}\mid a(A)\cdot\iota+0.1\cdot\mathbf{v}+0.1b\cdot\iota\in A\}\tag{6}$$

Monotonicity, *Strict Monotonicity* and maximality of $A\in\mathscr{L}(f)$ imply that $J_A(\mathbf{v})\in[-1,1]$. For

$\mathbf{z} \in \mathbb{R}^S$, denote by $\bar{z} = |S|^{-1} \cdot \sum_{s \in S} (\mathbf{z})_s$, $\mathbf{z}_\perp = \mathbf{z} - \bar{z} \cdot \iota$. Define $I : \mathbb{R}^S \to \mathbb{R}$ by

$$I_A(z) \equiv \begin{cases} \bar{z} + |\mathbf{z}_\perp| \cdot J_A\left(\dfrac{\mathbf{z}_\perp}{|\mathbf{z}_\perp|}\right) & \text{if} \quad \mathbf{z}_\perp \neq \mathbf{0} \\ \bar{z} & \text{if} \quad \mathbf{z}_\perp = \mathbf{0} \end{cases} \tag{7}$$

The following lemma is an analogue of Lemma 3.3. in Gilboa and Schmeidler (1989).

Lemma 7. *Let Axioms 1-7 hold. Then* $\forall f \in H_{0.2}$ $\forall A \in \mathcal{L}(f)$:

(a.i): $\forall \mathbf{z} \in H_{0.4}$ $\mathbf{z} \in A \iff I_A(\mathbf{z}) \leq \alpha(A)$;

(a.ii): $I_A(f) = \alpha(A)$;

(b.i): $I_A(\cdot)$ *is monotone:* $\mathbf{z} \geq \mathbf{z}' \implies I_A(\mathbf{z}) \geq I_A(\mathbf{z}')$;

(b.ii): $I_A(\cdot)$ *is positively homogeneous:* $I_A(\alpha \mathbf{z}) = \alpha I_A(\mathbf{z})$ *for all* $\alpha \geq 0$;

(b.iii): $I_A(\cdot)$ *is concave:* $I_A(\lambda \mathbf{z} + (1-\lambda)\mathbf{z}') \geq \lambda I_A(\mathbf{z}) + (1-\lambda) I_A(\mathbf{z}')$ *for all* $\lambda \in (0,1)$;

(b.iv): $I_A(\cdot)$ *is C-additive:* $I_A(\mathbf{z} + \beta \cdot \iota) = I_A(\mathbf{z}) + I_A(\beta \cdot \iota)$ *for all* $\beta \in \mathbb{R}$;

(b.v): $I_A(\cdot)$ *is normalized:* $I_A(\iota) = 1$.

Proof. Statements (b.ii), (b.iv), and (b.v) hold by the definition of I. Let us prove (a.i). Consider arbitrary $\mathbf{z} \in H_{0.4} = [0.3, 0.7]^S$. If $\mathbf{z}_\perp = 0$, $\mathbf{z} \in A \iff I_A(\mathbf{z}) \leq \alpha(A)$ by the definition of $\alpha(A)$. If $|\mathbf{z}_\perp| = 0.1$, $\mathbf{z} \in A \iff I_A(\mathbf{z}) \leq \alpha(A)$ by the definition of J. If $\bar{z} - \alpha(A) > |\mathbf{z}_\perp|$, then $\mathbf{z} >> (\alpha(A) + 0.5(\bar{z} - \alpha(A) - |\mathbf{z}_\perp|))\iota >> \alpha(A)\iota$. Thus $f \notin c(A \cup \{(\alpha(A) + 0.5(\bar{z} - \alpha(A) - |\mathbf{z}_\perp|))\iota\})$ by the definition of $\alpha(A)$, and by *Monotonicity*, $f \notin c(A \cup \{\mathbf{z}\})$, implying $\mathbf{z} \notin A$. Also, $I_A(\mathbf{z}) \geq \bar{z} - |\mathbf{z}_\perp| > \alpha(A)$, hence, $\mathbf{z} \in A \iff I_A(\mathbf{z}) \leq \alpha(A)$. If $\bar{z} - \alpha(A) < -|\mathbf{z}_\perp|$, then $\mathbf{z} \leq \alpha(A)\iota \in A$. Hence, $f \in c(A \cup \{\mathbf{z}\})$ by *Monotonicity*, and $\mathbf{z} \in A$ by maximality of A. Also, $I_A(\mathbf{z}) \leq \bar{z} + |\mathbf{z}_\perp| \leq \alpha(A)$, thus $\mathbf{z} \in A \iff I_A(\mathbf{z}) \leq \alpha(A)$.

Consider the last case $|\mathbf{z}_\perp| \neq 0, 0.1$ and $|\bar{z} - \alpha(A)| \leq |\mathbf{z}_\perp|$. Let $\mathbf{y} = \left(\alpha(A) + \frac{0.1\bar{z} - 0.1\alpha(A)}{|\mathbf{z}_\perp|}\right)\iota + 0.1\frac{\mathbf{z}_\perp}{|\mathbf{z}_\perp|}$. Note that $\mathbf{y} \in H_{0.2} \subset H$, $|\mathbf{y}_\perp| = 0.1$, and $\mathbf{y} + b\iota \in A \iff I_A(\mathbf{z}) \leq \alpha(A) - b$ for $b \in [-0.1, 0.1]$.

Suppose $|\mathbf{z}_\perp| < 0.1$, then $\mathbf{z} = \lambda \mathbf{y} + (1-\lambda)\alpha(A) \cdot \iota$, where $\lambda = 10|\mathbf{z}_\perp| \in (0,1)$. Let $I_A(\mathbf{z}) \leq \alpha(A)$, then $\{y, \alpha(A) \cdot \iota\} \subseteq A$, and by *No C-Hedging*, $f \in c(A \cup \{\mathbf{z}\})$, thus $\mathbf{z} \in A$ by maximality of A. Let $I_A(\mathbf{z}) > \alpha(A)$, consider $\epsilon = 0.01 \min\left\{1, \frac{1-\lambda}{\lambda}, \frac{I_A(\mathbf{z}) - \alpha(A)}{\lambda}\right\} > 0$, then $\mathbf{y} - \epsilon\iota, \left(\alpha(A) + \frac{\lambda}{1-\lambda}\epsilon\right)\iota \in H \setminus A$. By maximality of A, $f \notin c(A \cup \{\mathbf{y} - \epsilon\iota\})$, $f \notin c\left(A \cup \left\{\left(\alpha(A) + \frac{\lambda}{1-\lambda}\epsilon\right)\iota\right\}\right)$. Since $\mathbf{z} = \lambda(\mathbf{y} - \epsilon\iota) + (1-\lambda)\left(\alpha(A) + \frac{\lambda}{1-\lambda}\epsilon\right)\iota$, by *Indirect Ambiguity Aversion*, $f \notin c(A \cup \{\mathbf{z}\})$, hence $\mathbf{z} \notin A$.

Suppose $|\mathbf{z}_\perp| > 0.1$, then $\mathbf{y} = \lambda \mathbf{z} + (1-\lambda)\alpha(A) \cdot \iota$, where $\lambda = (10|\mathbf{z}_\perp|)^{-1} \in (0,1)$. Let $\mathbf{z} \in A$, then by *No C-Hedging*, $f \in c(A \cup \{\mathbf{y}\})$, hence $\mathbf{y} \in A$ by maximality of A, and $I_A(\mathbf{z}) \leq \alpha(A)$. Let $\mathbf{z} \notin A$; since A is closed, and $\mathbf{z} \in H_{0.4}$, $\exists \epsilon > 0$: $\mathbf{z} - \epsilon \cdot \iota, \left(\alpha(A) + \frac{\lambda}{1-\lambda}\epsilon\right)\iota \in H \setminus A$. By maximality of A, $f \notin c(A \cup \{\mathbf{z} - \epsilon\iota\})$ and $f \notin c\left(A \cup \left\{\left(\alpha(A) + \frac{\lambda}{1-\lambda}\epsilon\right)\iota\right\}\right)$. Since $\mathbf{y} = \lambda(\mathbf{z} - \epsilon\iota) + (1-\lambda)\left(\alpha(A) + \frac{\lambda}{1-\lambda}\epsilon\right)\iota$, by *Indirect Ambiguity*

Aversion, $f \notin c(A \cup \{\mathbf{y}\})$, hence $\mathbf{y} \notin A$, and $I_A(\mathbf{z}) > a(A)$, proving (a.i).

Let us prove (a.ii). If $I_A(f) > a(A)$, by (a.i), $f \notin A$, contradicting $f = c(A)$. If $I_A(f) < a(A)$, $\exists \epsilon > 0$: $\hat{f} = (1-\epsilon)f + \epsilon\iota >> f$, $I_A(\hat{f}) < a(A)$. Hence, $\hat{f} \in A$. By α, $f \in c(\{f, \hat{f}\})$, violating *Strict Monotonicity*.

Let us prove (b.i). If $\mathbf{z} = \mathbf{z}' = \mathbf{0}$, $I_A(\mathbf{z}) = 0 \geq 0 = I_A(\mathbf{z}')$. Otherwise, let $t = 0.01 \cdot \left(\max\{|z|, |z'|\}\right)^{-1}$, $\widetilde{\mathbf{z}} = t\mathbf{z} + (a(A) - I_A(t\mathbf{z}))\iota$, $\mathbf{w} = t\mathbf{z}' + (a(A) - I_A(t\mathbf{z}))\iota$. Then $I_A(\widetilde{\mathbf{z}}) = a(A)$, $\mathbf{w} \leq \widetilde{\mathbf{z}}$, and $\widetilde{\mathbf{z}}, \mathbf{w} \in H_{0.4}$. Hence, $\widetilde{\mathbf{z}} \in A$, by *Monotonicity*, $f \in c(A \cup \{\mathbf{w}\})$, by maximility of A, $\mathbf{w} \in A$, hence by (α), $I_A(\mathbf{w}) \leq a(A) = I_A(\widetilde{\mathbf{z}})$. By (b.ii), (b.iv), (b.v), $I_A(\mathbf{z}) \geq I_A(\mathbf{z}')$.

Let us prove (b.iii). If $\mathbf{z} = \mathbf{z}' = \mathbf{0}$, $I_A(\lambda\mathbf{z} + (1-\lambda)\mathbf{z}') = 0 = \lambda I_A(\mathbf{z}) + (1-\lambda)I_A(\mathbf{z}')$. Otherwise, let $t = 0.01 \cdot \left(\max\{|z|, |z'|\}\right)^{-1}$, $\tilde{\mathbf{z}} = t\mathbf{z} + (a(A) - I_A(t\mathbf{z}))\iota$, $\mathbf{q} = t\mathbf{z}' + \left(a(A) - I_A\left(t\mathbf{z}'\right)\right)\iota$. Let $\epsilon = 0.001$, then $\tilde{\mathbf{z}}, \mathbf{q} \in H_{0.4}$; by ($\alpha$), $\tilde{\mathbf{z}}, \mathbf{q} \notin A$, thus, $f \notin c(A \cup \{\tilde{\mathbf{z}} + \epsilon\iota\})$ and $f \notin c(A \cup \{\tilde{\mathbf{z}}' + \epsilon\iota\})$; by *Indirect Ambiguity Aversion*, $f \notin c(A \cup \{\lambda(\tilde{\mathbf{z}} + \epsilon\iota) + (1-\lambda)(\mathbf{q} + \epsilon\iota)\})$. By convexity of $H_{0.4}$, maximality of A, and other proven statements, $\lambda(\tilde{\mathbf{z}} + \epsilon\iota) + (1-\lambda)(\mathbf{q} + \epsilon\iota) \in H_{0.4} \backslash A \implies I_A(\lambda(\tilde{\mathbf{z}} + \epsilon\iota) + (1-\lambda)(\mathbf{q} + \epsilon\iota)) > a(A) \implies I_A(\lambda\tilde{\mathbf{z}} + (1-\lambda)\mathbf{q}) \geq a(A) \implies I_A\left(\lambda\left[t\mathbf{z} + (a(A) - I_A(t\mathbf{z}))\iota\right] + (1-\lambda)\left[t\mathbf{z}' + \left(a(A) - I_A\left(t\mathbf{z}'\right)\right)\iota\right]\right) \geq a(A)$ $\implies I_A\left(\lambda t\mathbf{z} + (1-\lambda)t\mathbf{z}'\right) \geq \lambda I_A(t\mathbf{z}) + (1-\lambda)I_A\left(t\mathbf{z}'\right) \implies I_A\left(\lambda\mathbf{z} + (1-\lambda)\mathbf{z}'\right) \geq \lambda I_A(\mathbf{z}) + (1-\lambda)I_A\left(\mathbf{z}'\right)$, proving that I is concave. □

Lemma 8. *Let $\mathscr{I}$ be the set of monotone positively homogeneous concave constant additive and normalized functions $I: \mathbb{R}^S \to \mathbb{R}$, and Π be the set of non-empty convex and closed sets of probabilities $P \subseteq \triangle S$. Then the mapping $\tau: \mathscr{I} \to \Pi$ given by $\tau(I) = \left\{\mu \in \triangle S \;\middle|\; \sum_{s \in S} \mu(s) z_s \geq I(\mathbf{z}) \; \forall \mathbf{z} \in \mathbb{R}^S\right\}$ is a bijection, and $\tau^{-1}(P)(\mathbf{z}) \equiv \min_{\mu \in P} \sum_{s \in S} \mu(s) z_s$. Moreover, $W_{\tau(I_A)}(f) = I_A(f) \; \forall f \in H$.*

Proof. This is a well-known statement. The details are given in the Online Appendix. □

Lemma 9. *If Axioms 1-7 hold for $c(\cdot)$, then $(U, cl(\mathscr{A}))$ represents $c(\cdot)$, where U is defined by Lemma 2, and $\mathscr{A} = \left\{\tau(I_B) \middle| f \in H_{0.2}, B \in \mathscr{L}(f)\right\}$.*

Proof. Let $f \in c(A)$, then $f_{0.1} \in c(A_{0.1})$. By Lemma 6, $\exists B \in \mathscr{L}(f_{0.1}): A_{0.1} \subseteq B$. By Lemmas 7,8, $I_B(f_{0.1}) = a(B) \geq I_B(g_{0.1}) \implies W_{\tau(I_B)}(f) = I_B(f) \geq I_B(g) = W_{\tau(I_B)}(g) \; \forall g \in A$. Finally, $\tau(I_B) \in \mathscr{A}$.

Let $P \in \mathscr{A}$ and let $W_P(g) \geq W_P(h) \; \forall h \in A$. Then $W_P(g_{0.1}) \geq W_P(h_{0.1}) \; \forall h_{0.1} \in A_{0.1}$. Hence, $\exists f \in H_{0.2} \; \exists B \in \mathscr{L}(f): P = \tau(I_B)$. Therefore, $I_B(g_{0.1}) \geq I_B(h_{0.1}) \; \forall h_{0.1} \in A_{0.1}$. Denote by $\lambda = \frac{5}{4}(a(B) - 0.2g_{0.1})$; since $0.4 \leq a(B) \leq 0.6$, $0.45 \leq g_{0.1} \leq 0.55$, then $\lambda \in [0.3625, 0.6375]$. For $h \in A$, let $\hat{h} = 0.2g_{0.1} + 0.8\lambda\iota$; for $\epsilon \in (0, 0.1)$, let $\hat{g}_\epsilon = 0.2g_{0.1} + 0.8(\lambda + \epsilon)\iota$. Then $I_B(\hat{h}) \leq I_B(\hat{g}) = a(B) < I_B(\hat{g}_\epsilon)$ $\forall h \in A \; \forall \epsilon \in (0, 1)$. Hence, $\hat{g}_\epsilon \notin B$; by maximality of B, $f \notin c(B \cup \{\hat{g}_\epsilon\})$; by *Aizerman's Property*, $\hat{g}_\epsilon \in c(B \cup \{\hat{g}_\epsilon\})$; by *Act Continuity*, $\hat{g} \in c(B \cup \{\hat{g}\}) = c(B)$. By maximality of B, $0.2A_{0.1} + 0.8\lambda\iota \subseteq B$;

by α, $\hat{g} \in c(0.2A_{0.1} + 0.8\lambda\iota)$; by *C-Independence*, $g_{0.1} \in c(A_{0.1})$, $g \in c(A)$.

Claim 4. *If $P \in cl(\mathscr{A})\backslash\mathscr{A}$ and $W_P(g) \geq W_P(h)$ $\forall h \in A$, then $g \in c(A)$.*

Proof of Claim 4. Consider $P^k \longrightarrow P$, $P^k \in \mathscr{A}$. Let $g^n_{0.1} = \left(1 - \frac{1}{n}\right) g_{0.1} + \frac{1}{n}\iota$, then $W_P(g^n_{0.1}) > W_P(g_{0.1}) \geq W_P(h_{0.1})$ $\forall h \in A$. By continuity of W, $W_{P^{k(n)}}(g^n_{0.1}) \geq W_P(h_{0.1})$ $\forall h \in A$ for $k(n)$ large enough. By the proof in the paragraph above, $g^n_{0.1} \in c(A_{0.1} \cup \{g^n_{0.1}\})$. By *Act Continuity*, $g_{0.1} \in c(A_{0.1} \cup \{g_{0.1}\}) = c(A_{0.1})$. By *C-Independence*, $g \in c(A)$, proving the Claim and the Lemma. □

Lemmas 5, 9 prove the "if" direction of the Theorem. ■

Proof of Proposition 1

For each of the axioms, we give examples when the corresponding axiom is violated, while other axioms hold. For brevity, the proofs that other axioms hold are omitted; an interested reader can find them in the Online Appendix.

Framed Uncertainty. Let $\{x, y, z\} \subseteq X$. Consider

$$c(A) = \left\{ f \in A \;\middle|\; \sum_{s \in S} f(s)(x) \geq \sum_{s \in S} g(s)(x) \;\forall g \in A \text{ or } \sum_{s \in S} f(s)(y) \geq \sum_{s \in S} g(s)(y) \;\forall g \in A \right\}$$

Since $c(\{x, 0.5x+0.5y\}) = \{x, 0.5x+0.5y\}$ and $c(\{x, 0.5x+0.5y, y\}) = \{x, y\}$, C-β, and, hence, *Framed Uncertainty* is violated. Other axioms hold; in particular, $z \notin c(\{x, z\})$, so *C-Non-Degeneracy* holds.

C-Independence. Let $\{x, y\} \subseteq X$, $c(A) = \arg\max_{f \in A} W(f)$, where $W(f) = \dfrac{f(1)(x) + f(2)(x)}{1 + f(2)(x)}$. Let $f = (0.4x + 0.6y, y, \ldots)$ and $g = (y, x, \ldots)$. Then $W(f) = 0.4 < 0.5 = W(g)$, hence $c(\{f, g\}) = \{g\}$. Next, consider $f' = 0.5f + 0.5x = (0.7x + 0.3y, 0.5x + 0.5y, \ldots)$ and $g' = 0.5g + 0.5x = (0.5x + 0.5y, x, \ldots)$. Then $W(f') = 1.2/1.5 = 4/5 > 3/4 = 1.5/2.0 = W(g')$, hence $c(0.5\{f, g\} + 0.5\{p\}) = \{f'\} = \{0.5f + 0.5p\}$, contradicting *C-Independence*. Other axioms hold.

No-C-Hedging. Let $x \in X$, $0 < \underline{\mu} \leq \overline{\mu} \leq 1$. Consider $c(B) = \bigcup_{k \in [0,\infty)} \arg\max_{f \in B} W_k(f)$, where

$$W_k(f) = \min\left\{(1 - \overline{\mu}) f(1)(x) + \overline{\mu} f(2)(x), (1 - \underline{\mu}) f(1)(x) + \underline{\mu} f(2)(x), f(1)(x) + \underline{\mu} k\right\}$$

Consider $f, g \in H$, $p \in H_0$ with $f(1)(x) = 0.2$, $f(2)(x) = 0.3$, $g(1)(x) = 0.2$, $g(2)(x) = 1$, $p(x) = 0.2 + 0.1\underline{\mu}$. Then $W_k(f) = 0.2 + \underline{\mu} \cdot \min\{0.1, k\}$, $W_k(g) = 0.2 + \underline{\mu} \cdot \min\{1, k\}$, $W_k(p) = 0.2 + \underline{\mu} \cdot 0.1$, and $W_k(0.5g + 0.5p) = 0.2 + \underline{\mu} \cdot \min\{0.45, k + 0.05\} > W_k(f)$. Hence, $k = 0.1$ justifies the choice of f from the menu $A = \{f, g, p\}$, but f is not the best in the menu $A \cup \{0.5g + 0.5p\}$ for any $k \in [0, \infty)$. Thus, *No-C-Hedging* fails. Other axioms hold.

Strict Monotonicity. Let $x \in X$, $c(A) = \arg\max_{f \in A} W(f)$, $W(f) = 2f(1)(x) - f(2)(x)$. Consider

$f,g \in H$ with $f(1)(x) = f(2)(x) = 0.3$, $g(1)(x) = 0.4$, $g(2)(x) = 0.7$. Then $W(f) = W(f(1)) = W(f(2)) = 0.3$, and $W(g) = 0.1$, $W(g(1)) = 0.4$, $W(g(2)) = 0.7$. Hence, the pair f,g violates *Strict Monotonicity*. Other axioms hold.

Indirect Ambiguity Aversion. Let $x \in X$, $c(A) = \arg\max_{f\in A} W(f)$, $W(f) = \max\{f(1)(x), f(2)(x)\}$. Consider $h \in H_0$, $f,g \in H$ with $h(x) = 0.7$, $f(1)(x) = 1$, $f(2)(x) = 0$, $g(1)(x) = 0$, $g(2)(x) = 1$, and $A = \{h\}$. Then $h \notin c(A \cup \{f\})$, $h \notin c(A \cup \{g\})$, but $h \in c(A \cup \{0.5f + 0.5g\})$, violating *Indirect Ambiguity Aversion*. Other axioms hold.

Continuity. Let $x \in X$, $c(A) = \{f \in A | f \geq g\ \forall g \in A\}$, where $f \geq g$ if either $f(1)(x) > g(1)(x)$, or $[f(1)(x) = g(1)(x)$ and $f(2)(x) \geq g(2)(x)]$. Consider $f, g, f_n \in H$, $n = 1,2,..$ with $g(1)(x) = g(2)(x) = 0.5$, $f_n(1)(x) = 0.5 + 1/n$, $f_n(2)(x) = 0$, $f = \lim_{n\to\infty} f_n$ (pick f_n such that the limit exists). Then $f_n \in c(\{f_n, g\})$, but $f \notin c(\{f, g\})$, violating *Continuity*. Other axioms hold.

C-Non-Degeneracy. Consider $c(A) = A$; *C-Non-Degeneracy* fails, and other axioms hold. ■

Proof of Theorem 2

By Lemma 2, U is unique up to the positive affine transformation. By Lemma 5, WLOG, $X = \{x^*, x_*\}$; normalizing $U(x^*) = 1$, $U(x_*) = 0$, we get $H = [0,1]^S$, $W_P(\mathbf{z}) = \tau^{-1}(P)(\mathbf{z})$ (by Lemma 8).

Lemma 10. *If $(U, \mathscr{A}')$ represents $c(\cdot)$, then $\mathscr{A} \subseteq \mathscr{A}'$, where $\mathscr{A} = \left\{\tau(I_B) \middle| f \in H_{0.2}, B \in \mathscr{L}(f)\right\}$.*

Proof. Consider arbitrary $f \in H_{0.2}$, $A \in \mathscr{L}(f)$. Let $\mathscr{C}' = \{P' \in \mathscr{A}' | A \subseteq B(P')\}$, where $B(P') = \{\mathbf{z} \in H | \tau^{-1}(P')(f) \geq \tau^{-1}(P')(\mathbf{z})\}$. Since $f \in c(A)$, then $\mathscr{C}' \neq \varnothing$. If $B(P') \neq A$ for $P' \in \mathscr{C}'$, then $f \in c(B(P'))$ (note that $B(P')$ is closed, hence $B(P') \in \mathscr{K}$), contradicting maximality of A. Thus, $B(P') = A$ for all $P' \in \mathscr{C}'$, and $\tau^{-1}(P')(f) = a(A) = I_A(f)$. Consider arbitrary $\mathbf{z} \in \mathbb{R}^S$. If $\mathbf{z} = 0$, then $I_A(\mathbf{z}) = 0 = \tau^{-1}(P')(\mathbf{z})$. Otherwise, $\mathbf{w} = 0.1\frac{\mathbf{z}}{|\mathbf{z}|} + 0.5\iota \in H_{0.2}$, and $\mathbf{w} + 0.4b\iota \in H\ \forall b \in [-1,1]$. Also, $\mathbf{w} - 0.4\iota << 0.4\iota \leq a(A)\iota \leq 0.6\iota << \mathbf{w} + 0.4\iota$. Then $I_A(\mathbf{w} - 0.4\iota), \tau^{-1}(P')(\mathbf{w} - 0.4\iota) < a(A) < I_A(\mathbf{w} + 0.4\iota), \tau^{-1}(P')(\mathbf{w} + 0.4\iota)$. Moreover, since $B(P') = A$, $I_A(\mathbf{w} + 0.4b\iota) \geq a(A) \iff \tau^{-1}(P')(\mathbf{w} + 0.4b\iota) \geq a(A)\ \forall b \in [-1,1]$. Therefore, $\exists b \in [-1,1]$: $I_A(\mathbf{w} + 0.4b\iota) = a(A) = \tau^{-1}(P')(\mathbf{w} + 0.4\mathbf{b}\iota)$. Thus, $I_A(\mathbf{z}) = \tau^{-1}(P')(\mathbf{z})\ \forall \mathbf{z} \in \mathbb{R}^S$, hence $\tau(I_A) = P' \in \mathscr{A}'$. □

Consider any representation $(U, \mathscr{A}')$. By Lemma 10, $cl(\mathscr{A}) \subseteq cl(\mathscr{A}') = \mathscr{A}'$; by Lemma 9, $(U, cl(\mathscr{A}))$ represents $c(\cdot)$. Hence, $cl(\mathscr{A})$ is the minimum family of frames. Let $\mathscr{B} = \left\{P \in \Pi \middle| \forall A \in \mathscr{K}\, \forall f \in A\left[W_P(f) \geq W_P(g) \forall g \in A \implies f \in c(A)\right]\right\}$. By argument identical to the one used in the proof of Claim 4, we get $cl(\mathscr{B}) = \mathscr{B}$, hence $(U, \mathscr{B})$ is a framed ambiguity model; let c'' be its induced choice correspondence. Since $cl(\mathscr{A}) \in cl(\mathscr{B}) = \mathscr{B}$, then $c \subseteq c''$; by the definition of $\mathscr{B}$ we also have $c'' \subseteq c$ and $\mathscr{A}' \in \mathscr{B}$. Therefore, $\mathscr{B}$ is the maximum family of frames. Next, let $\mathscr{D}$ be a closed family of frames such that $cl(\mathscr{A}) \subseteq \mathscr{D} \subseteq \mathscr{B}$, and let c' be the choice correspondence represented by $(U, \mathscr{D})$. Then $c \subseteq c' \subseteq c'' = c$, hence $c' = c$, proving Theorem 2 except of the last statement.

Lemma 11. *There exists a coherent intersection P of a family $\mathscr{C}$ if and only if*

$$I_P(z) = \max_{P' \in \mathscr{C}} I_{P'}(z) \;\; \forall z \in \mathbb{R}^S, \quad \textit{where } \; I_Q(z) = \tau^{-1}(Q)(z) = \min_{\mu \in Q} \sum_{s \in S} \mu(s) \cdot z_s$$

Proof. Let P be a coherent intersection of $\mathscr{C}$. For an arbitrary $z \in \mathbb{R}^S$, let t be its component orthogonal to $(1, \ldots, 1)$. If $t = 0$, $I_{\tilde{P}}(z)$ does not depend on $\tilde{P}$. Otherwise, the projections of P and $\mathscr{C}$ on the linear subspace $\mathbb{T}_t = \{x \in \mathbb{R}^S \mid \exists \lambda \in \mathbb{R} : \; x = \lambda t\}$ are an interval $[a_t(P), b_t(P)]$ and a collection of intervals $\{[a_t(P'), b_t(P')]\}_{P' \in \mathscr{C}}$. Then, WLOG, $I_{P'}(t) = a_t(P')$ for all $P' \in \mathscr{C} \cup P$. Since P is a coherent intersection, then $[a_t(P), b_t(P)] = \bigcap_{P' \in \mathscr{C}} [a_t(P'), b_t(P')] \neq \emptyset$. It follows that

$$I_P(z) - \sum_{i=1}^{S} z_s = I_P(t) = a_t(P) = \max_{P' \in \mathscr{C}} a_t(P') = \max_{P' \in \mathscr{C}} I_{P'}(t) = \max_{P' \in \mathscr{C}} I_{P'}(t) - \sum_{i=1}^{S} z_s$$

proving the only if direction of Lemma 11.

Claim 5. *(i) $P \subseteq P'$ if and only if $I_P(z) \geq I_{P'}(z)$ for all $z \in \mathbb{R}^S$, (ii) $P \subsetneq P'$ if and only if $I_P(z) \geq I_{P'}(z)$ for all $z \in \mathbb{R}^S$ and $\exists \hat{z} \in \mathbb{R}^S : I_P(\hat{z}) > I_{P'}(\hat{z})$.*

Proof Claim 5. The claim follows from Lemma 8. □

Claim 6. *If $I_P(z) = \max_{P' \in \mathscr{C}} I_{P'}(z)$ for all $z \in \mathbb{R}^S$, then $P = \bigcap_{P' \in \mathscr{C}} P' \neq \emptyset$.*

Proof Claim 6. Since $\forall P' \in \mathscr{C}$, $I_P(z) = \max_{P' \in \mathscr{C}} I_{P'}(z) \geq I_{P'}(z) \forall z \in \mathbb{R}^S$, by Claim 5, $P \subseteq P'$ for all $P' \in \mathscr{C}$, hence $P \subseteq \bigcap_{P' \in \mathscr{C}} P' \neq \emptyset$. If $P \neq \tilde{P} = \bigcap_{P' \in \mathscr{C}} P'$, then by Claim 5, $\exists z \in \mathbb{R}^S : I_P(z) > I_{\tilde{P}}(z) \geq \max_{P' \in \mathscr{C}} I_{P'}(z)$, in contradiction. □

Claim 7. *For any linear subspace $\mathbb{T}$ of $\mathbb{R}^S$, $proj_{\mathbb{T}}\left(\bigcap_{P' \in \mathscr{C}} P'\right) \subseteq \bigcap_{P' \in \mathscr{C}} proj_{\mathbb{T}}(P')$.*

Proof Claim 7. If $\hat{x} \in proj_{\mathbb{T}}\left(\bigcap_{P' \in \mathscr{C}} P'\right)$, then $\exists x \in P : \; \hat{x} = proj_{\mathbb{T}}(x)$. Hence, $x \in P'$ for all $P' \in \mathscr{C}$ and $\hat{x} \in \bigcap_{P' \in \mathscr{C}} proj_{\mathbb{T}}(P')$. □

Assume $I_P(z) = \max_{P' \in \mathscr{C}} I_{P'}(z)$. Towards a contradiction, assume $proj_{\mathbb{T}}\left(\bigcap_{P' \in \mathscr{C}} P'\right) \subsetneq \bigcap_{P' \in \mathscr{C}} proj_{\mathbb{T}}(P')$ for some linear subspace $\mathbb{T}$. By the Separating Hyperplane Theorem applied to the closed convex set $\hat{P} = proj_{\mathbb{T}}\left(\bigcap_{P' \in \mathscr{C}} P'\right)$ and point $\hat{\mu} \in \bigcap_{P' \in \mathscr{C}} proj_{\mathbb{T}}(P') \setminus \hat{P} \neq \emptyset$, there is a vector $\hat{t} \in \mathbb{T}^* = \mathbb{T}$ such that $\min_{\hat{\nu} \in \hat{P}} \sum_i \hat{t}_i \hat{\nu}(i) > \sum_i \hat{t}_i \hat{\mu}(i) \geq \max_{\hat{P}' \in proj_{\mathbb{T}}(\mathscr{C})} \min_{\hat{\nu} \in \hat{P}} \sum_i \hat{t}_i \hat{\nu}(i)$. Since $\theta_{\hat{t}} : \mathbb{R}^S \to \mathbb{R}$ given by $\theta_{\hat{t}}(z) = \sum_i \hat{t}_i \left(proj_{\mathbb{T}}(z)\right)_i$ is a linear function, there is $t \in \mathbb{R}^S$ such that $\theta_{\hat{t}}(z) = \sum_{s \in S} t_s z_s$; then,

$I_P(z) < \max_{P' \in \mathscr{C}} I_{P'}(z)$, in contradiction. Lemma 11 is proven. $\square$

Suppose that P is a coherent intersection of a closed collection $\mathscr{C} \subseteq \mathscr{A}$, where $\mathscr{A}$ is the minimum family of frames. Consider arbitrary $A \in \mathscr{K}$ and $f \in A$ such that $W_P(f) \geq W_P(g)$ for all $g \in A$. By Lemma 11, there is $\tilde{P} \in \mathscr{C}$ such that $W_{\tilde{P}}(f) = W_P(f) \geq W_P(g) = \max_{P' \in \mathscr{C}} W_{P'}(g) \geq W_{\tilde{P}}(g)$ for all $g \in A$. Since $\tilde{P} \in \mathscr{A}$, $f \in c(A)$; hence, $P \in \mathscr{B}$, where $\mathscr{B}$ is the maximum family of frames.

Conversely, suppose $P \in \mathscr{B}$; thus, $[W_P(f) \geq W_P(g) \forall g \in A \implies f \in c(A)]$. Let

$V = \left\{ v \in \mathbb{R}^S \middle| \sum_{s \in S} v_s = 0, \max_{s \in S} |v_s| = 1 \right\}$, $B = \left\{ z \in H \middle| W_P(z) \geq 0.5 \right\}$, $z^v = (0.5 - 0.01 W_P(v)\iota) + 0.01v$

for $v \in V$. Note that $z^v \in H$ and $W_P(z^v) = 0.5$. Hence, $z^v \in B$, and $\forall v \in V \exists P^v \in \mathscr{A}$ such that $W_{P^v}(z^v) \geq W_{P^v}(z) \forall z \in B$. Since $0.5\iota \in B$, $W_{P^v}(z^v) \geq W_{P^v}(0.5\iota) = 0.5$. Towards a contradiction, assume $W_{P^v}(z^v) > 0.5$. Consider $g = (0.5 - 0.02 W_P(v)) + 0.02v \in H$; since $W_P(g) = 0.5$, $g \in B$. However, $z^v = 0.5g + 0.5 \cdot 0.5\iota$, hence $W_{P^v}(z^v) = 0.5 W_{P^v}(g) + 0.25$, and $W_{P^v}(g) = W_{P^v}(z^v) + (W_{P^v}(z^v) - 0.5) > W_{P^v}(z^v)$, contradicting the definition of P^v. Therefore, $W_{P^v}(z^v) = 0.5 \geq W_{P^v}(z^{v'})$ for all $v, v' \in V$. Since $W_{\tilde{P}}$ is positively homogeneous, and $W_{\tilde{P}}(0.5\iota) = 0.5$ for all $\tilde{P} \in \Pi$, for any $z \in \mathbb{R}^S$ there exists $P^z \in \mathscr{A}$ such that $W_P(z) = W_{P^z}(z)$, and $W_P(z') \geq W_{P^z}(z')$ for all $z' \in \mathbb{R}^S$; denote by $\mathscr{C}^z \neq \varnothing$ the collection of all frames in $\mathscr{A}$ that has this property, and let $\mathscr{C} = \bigcup_{z \in Z} \mathscr{C}^z$. Then

$$W_P(z) = \max_{P' \in \mathscr{C}} W_{P'}(z) \quad \forall z \in \mathbb{R}^S$$

Since $W_{\tilde{P}}$ is continuous with respect to $\tilde{P}$, $cl(\mathscr{C}) = \mathscr{C}$. Therefore, P is the coherent intersection of a closed collection of frames $\mathscr{C} \subseteq \mathscr{A}$, where $\mathscr{A}$ is the minimum family of frames, proving the last assersion of the Theorem. $\blacksquare$

Proof of Proposition 2

The proof that $|\mathscr{A}| = 1$ implies all of the considered properties is straightforward and omitted. Assume $|\mathscr{A}| > 1$. Then there are $P_1, P_2 \in \mathscr{A}$, $f \in H$, and $q_1, q_2 \in H^0$ such that $U(q_1) = W_{P_1}(f) > W_{P_2}(f) = U(q_2)$. Note that $c(\{q_2, f\}) = \{q_2, f\}$, but $c(\{q_1, q_2, f\}) = \{q_1, f\}$, thus β is violated, and, therefore, *WARP is violated.*

Consider $g = 0.25q_1 + 0.75q_2$, $h = 0.5f + 0.5g = 0.5f + 0.125q_1 + 0.375q_2$, $w = 0.5f + 0.25q_1 + 0.25q_2$. Then $W_{P_1}(f) > W_{P_1}(w) > W_{P_1}(h) > W_{P_1}(g)$ and $W_{P_2}(g) = W_{P_2}(w) > W_{P_2}(h) > W_{P_2}(f)$. Hence, $f, g \in c(\{f, g, w, 0.5f + 0.5g\})$. Since $w >> h$, $h \notin c(\{f, g, w, 0.5f + 0.5g\})$, violating *Ambiguity Aversion.* Considering $A = \{w\}$ shows the violation of *Direct Ambiguity Aversion* as well.

Consider $p = 0.5q_1 + 0.5q_2 \in H_0$. Since $W_{P_2}(h) > W_{P_2}(f)$, $h \in c(\{h, f\})$. Since $W_{P_1}(h) > W_{P_1}(p)$,

$h \in c(\{h,p\})$. Since $0.5f + 0.5p = w >> h$, $h \notin c(\{h, 0.5f + 0.5p\})$, violating *Pairwise No-C-Hedging*.

Towards a contradiction, assume *Normality* holds. Let $f, h \in H$, $p \in H^0$, $\lambda \in (0,1)$ be such that $h \in c(\{h,f\})$ and $h \in c(\{h,p\})$. By *Normality*, $h \in c(\{h,f,p\})$, by *No-C-Hedging*, $h \in c(\{h,f,p,\lambda f + (1-\lambda)p\})$, by α, $h \in c(\{h, \lambda f + (1-\lambda)p\})$; thus, *Pairwise No-C-Hedging* holds, in contradiction. Finally, α and γ are equivalent to *Normality*, hence γ is violated as well. ■

Proof of Proposition 3

If U_1 is a positive affine transformation of U_2 and $\mathscr{A}_1 \subseteq (\mathscr{A}_2)_{coh}$, then for all $B \in \mathscr{K}$ we have

$$c_1(B) = \bigcup_{P \in \mathscr{A}_1} \arg\max_{f \in B} W_P(f) \subseteq \bigcup_{P \in (\mathscr{A}_2)_{coh}} \arg\max_{f \in B} W_P(f) = c_2(B)$$

Hence, DM 1 is more decisive than DM 2. Conversely, assume DM 1 is more decisive than DM 2.

Lemma 12. *Let c_1 and c_2 have framed ambiguity representations $(U_1, \mathscr{A}_1)$ amd $(U_2, \mathscr{A}_2)$. Assume that for all $f, g \in H$ we have $c_2(\{f,g\}) = \{f\} \implies c_1(\{f,g\}) = \{f\}$. Then U_1 and U_2 are positive affine transformations of each other.*

Proof. Since U_2 is nondegenerate, $\exists x, y \in \triangle X : U_2(x) > U_2(y)$; then $c_2(\{x,y\}) = \{x\}$, $c_1(\{x,y\}) = \{x\}$, and $U_1(x) > U_1(y)$. Similarly, $\forall p, q \in \triangle X$, if $U_2(p) > U_2(q)$, then $U_1(p) > U_1(q)$. Suppose now $U_2(p) = U_2(q)$. Let

$$\alpha = 0.01 \cdot \left(\max\{|U_2(p)|, |U_2(y)|, 1\}\right)^{-1}, \quad \beta = \frac{0.5[U_2(x) - U_2(y)] - \alpha(U_2(p) - U_2(y))}{(1-\alpha)(U_2(x) - U_2(y))}$$

then $\alpha, \beta \in (0,1)$. For $\gamma \in (0,1)$, define $p^{\alpha}_{\gamma} = \alpha p + (1-\alpha)(\beta x + (1-\beta)y)$, $q^{\alpha}_{\gamma} = \alpha q + (1-\alpha)(\beta x + (1-\beta)y)$, then for $\epsilon > 0$ small enough, $\beta - \epsilon, \beta + \epsilon \in (0,1)$, and $U_2(p^{\alpha}_{\beta-\epsilon}) < 0.5U_2(x) + 0.5U_2(y) < U_2(q^{\alpha}_{\beta+\epsilon})$. It follows $U_1(p^{\alpha}_{\beta-\epsilon}) < 0.5U_1(x) + 0.5U_1(y) < U_1(q^{\alpha}_{\beta+\epsilon})$. By continuity of U_1, $U_1(p^{\alpha}_{\beta}) = U_1(q^{\alpha}_{\beta})$, which implies $U_1(p) = U_1(q)$. Hence, U_2 and U_1 represent the same linear preferences on $\triangle X$, and they are positive affine transformations of each other. □

By Lemma 12, WLOG, $U_1 = U_2$. Since $P \in \mathscr{A}_1 \implies [f \in A, W_P(f) \geq W_P(g) \forall g \in A \implies f \in c(A)]$, then by Theorem 2, $P \in (\mathscr{A}_2)_{coh}$. ■

Proof of Theorem 3

Throughout the proof $I_P(\cdot) = \tau^{-1}(P)(\cdot)$ is the support functional of set of beliefs P defined in (8). We'll first prove the supporting lemmas, and then $(i) \Longrightarrow (iii) \Longrightarrow (ii) \Longrightarrow (i)$.

For a fixed (up to a positive affine transformation) vNM expected utility function U and $P \in \Pi$, let W_P be the associated maxmin expected utility function and $f \succeq_P g$ iff $W_P(f) \geq W_P(g)$; $f \succ_P g$ iff $f \succeq_P g$ and $g \not\succeq_P f$.

Lemma 13. *Let $c_1(\cdot)$ and $c_2(\cdot)$ have framed ambiguity representations $(U_1, \mathscr{A}_1)$ and $(U_2, \mathscr{A}_2)$. Then the following statements are equivalent:*
(i) For all $A \in \mathscr{K}$ $\quad |c_2(A)| = 1 \implies |c_1(A)| = 1$;
(ii) For all $A \in \mathscr{K}$ and $f \in H$ $\quad c_2(A) = \{f\} \implies c_1(A) = \{f\}$;
(iii) For all $f, g \in H$ $\quad c_2(\{f,g\}) = \{f\} \implies c_1(\{f,g\}) = \{f\}$;
(iv) U_1 is a positive affine transformation of U_2, and $[f \succeq_P g\ \forall P \in \mathscr{A}_2] \Longrightarrow [f \succeq_P g\ \forall P \in \mathscr{A}_1]$;
(v) U_1 is a positive affine transformation of U_2, and $[f \succ_P g\ \forall P \in \mathscr{A}_2] \Longrightarrow [f \succ_P g\ \forall P \in \mathscr{A}_1]$.

Proof. Implications (ii) $\implies$ (i), (iii) are straightforward. Let (iii) hold. Towards a contradiction, assume $c_2(A) = \{f\}$, and $g \in c_1(A) \backslash \{f\}$. If $g \in c_2(\{g, f\})$, $\exists P \in \mathscr{A}_2$ such that $W_P(g) \geq W_P(f)$, then $\{f\} \neq \arg\max_{h \in A} W_P(h)$ and $c_2(A) \neq \{f\}$. Thus, $c_2\{g, f\} = \{f\}$. By α, $g \in c_1(\{g, f\})$, contradicting (iii). Since $c_1(A) \neq \varnothing$, $c_1(A) = \{f\}$. Hence, (iii) $\Longrightarrow$ (ii). Now, let us prove that (i) implies (iii).

Towards a contradiction, assume $c_2(\{f, g\}) = \{f\}$, but $c_1(\{f, g\}) = \{g\}$ for some $f, g \in H$. Since U_1 is non-degenerate, $\exists p, q$: $U_1(p) > U_1(q)$. Let $\hat{f} = 0.5f + 0.25p + 0.25q$, $\hat{g} = 0.5g + 0.25p + 0.25q$. By *C-Independence*, $c_2(\{\hat{f}, \hat{g}\}) = \{\hat{f}\}$, $c_1(\{\hat{f}, \hat{g}\}) = \{\hat{g}\}$. Given $P \in \Pi$, let W_P^1 and W_P^2 are maxmin expected utility functions associated with U_1 and U_2. Denote by $\epsilon_1 = \inf_{P \in \mathscr{A}_1} \left(W_P^1(\hat{g}) - W_P^1(\hat{f})\right)$, $\epsilon_2 = \inf_{P \in \mathscr{A}_2} \left(W_P^2(\hat{f}) - W_P^2(\hat{g})\right)$. Note that W^1, W^2 are continuous in P, hence, $W_P^1(f) - W_P^1(g)$ and $W_P^2(f) - W_P^2(g)$ are uniformly continuous in P on compact sets $\mathscr{A}_1$ and $\mathscr{A}_2$. Therefore, since $W_P^1(\hat{g}) - W_P^1(\hat{f}) > 0\ \forall P \in \mathscr{A}_1$ and $W_P^2(\hat{f}) - W_P^2(\hat{g}) > 0 \forall P \in \mathscr{A}_2$, then $\epsilon_1, \epsilon_2 > 0$. For $\delta \in (0,1)$, consider

$$A = \left\{0.5g + 0.5(1-\delta)(0.5p + 0.5q) + 0.5\delta h \middle| \exists \lambda \in [0,1]^S : h(s) = \lambda_s p + (1 - \lambda_s) q\right\}$$

Since $\epsilon_1, \epsilon_2 > 0$, $\exists \delta > 0$: $W_P^1(\hat{f}) < W_P^1(h) \forall h \in A \forall P \in \mathscr{A}_1$, $W_P^2(\hat{f}) > W_P^2(h) \forall h \in A \forall P \in \mathscr{A}_2$. Let $Q \in \mathscr{A}_1 \neq \varnothing$ and $B = \{h \in A | W_Q^1(h) = W_Q^1(\hat{g})\}$. Since $\hat{g} \in B$ and $|S| > 1$ then $|B| > 1$. Hence, $|c_1(B \cup \{\hat{f}\})| = |B| > 1$ and $|c_2(B \cup \{\hat{f}\})| = |\{f\}| = 1$, contradicting (i); hence, (i) $\Longrightarrow$ (iii).

Let (iii) hold, then by Lemma 12, U_1 is a positive affine transformation of U_2. Next, $[f \succeq_P g\ \forall P \in \mathscr{A}_2] \Longleftrightarrow [c_2(\{f, g\}) = \{f\}] \implies [c_1(\{f, g\}) = \{f\}] \Longleftrightarrow [f \succ_P g\ \forall P \in \mathscr{A}_1]$. Hence, (v) holds. Conversely, suppose (v) holds, then $[c_2(\{f, g\}) = \{f\}] \Longleftrightarrow [f \succeq_P g\ \forall P \in \mathscr{A}_2] \implies [f \succ_P g\ \forall P \in \mathscr{A}_1] \Longleftrightarrow [c_1(\{f, g\}) = \{f\}]$, proving (iii). Hence, (iii)$\Longleftrightarrow$(v)

Let (v) hold. WLOG, $U_1 = U_2 = U$; let p, q be such that $U(p) > U(q)$. Denote by $\hat{f}^n = 0.5f + 0.5(1 - 1/n)(0.5p + 0.5q) + (0.5/n)p$, $\hat{f} = 0.5f + 0.5(0.5p + 0.5q)$, $\hat{g} = 0.5g + 0.5(0.5p + 0.5q)$. Since $U(\hat{f}^n(s)) = U(\hat{f}(s)) + (U(p) - U(q))/4n > U(\hat{f}^n(s))\ \forall s \in S$, and $W_P(h) = 0.5W_P(h) + 0.25U(p) +$

$0.25U(q)$, then $[f \succeq_P g\ \forall P \in \mathscr{A}_2] \implies [\hat{f} \succeq_P \hat{g}\ \forall P \in \mathscr{A}_2] \implies [\hat{f}^n \succ_P \hat{g}\ \forall P \in \mathscr{A}_2] \implies [\hat{f}^n \succ_P \hat{g}\ \forall P \in \mathscr{A}_1]$. Since $W_P(\cdot)$ is continuous, $[\hat{f} \succeq_P \hat{g}\ \forall P \in \mathscr{A}_1]$ and $[f \succeq_P g\ \forall P \in \mathscr{A}_1]$, proving (iv).

Let (iv) hold. Assume $[f \succ_P g\ \forall P \in \mathscr{A}_2]$. Note the function $\zeta : \Pi \times H \times H$ given by $\zeta(P,f,g) = W_P(f) - W_P(g)$ is uniformly continuous on the compact set $\mathscr{A}_2 \times H \times H$. Therefore, $\exists \epsilon > 0$: $W_P(f) > W_P(g) + 2\epsilon\ \forall P \in \mathscr{A}_2$. Using $\hat{f}$, $\hat{g}$ defined in the paragraph above, we get $W_P(\hat{f}) > W_P(\hat{g}) + \epsilon$ $\forall P \in \mathscr{A}_2$. Let $\hat{f}_\gamma = 0.5f + 0.5(1-\gamma)(0.5p + 0.5q) + 0.5\gamma q$, then for $\gamma > 0$ small enough, $W_P(\hat{f}) > W_P(\hat{f}_\gamma) > W_P(\hat{f}) - \epsilon/2 > W_P(\hat{g})\ \forall P \in \mathscr{A}_2$. By (iv) and the fact that $U_1(\hat{f}(s)) > U_1(\hat{f}_\gamma(s))\ \forall s \in S$, we have $W_P(\hat{f}) > W_P(\hat{f}_\gamma) \geq W_P(\hat{g})\ \forall P \in \mathscr{A}_1$. Therefore, $W_P(f) > W_P(g)\ \forall P \in \mathscr{A}_1$, proving (iv)$\implies$(v).

Since (iv)$\iff$(v) $\iff$(iii)$\implies$ (ii)$\implies$ (i)$\implies$ (iii), all statements of Lemma 13 are equivalent. $\square$

The following lemma is closely related to lemmas 1-7 in Crès et al. (2011). The difference is that $\succeq_P$ does not satisfy EUA axiom from Crès et al. (2011) with respect to $\succeq_{P_i}$, but only the Unanimity axiom. We also borrowed some of the ideas of proofs from their lemmas 1-7. Let $X = \{x,y\}$, $U(x) = 1$, $U(y) = -1$, $H = [-1,1]^S$; denote by $W_P(f) = \min_{\mu \in P} \sum_{s \in S} f_s \mu(s)$ for $f \in \mathbb{R}^S$.

Lemma 14. *Let $X = \{x,y\}$, $U(x) = 1$, $U(y) = -1$, $H = [-1,1]^S \subset \mathbb{R}^S$, $\mathscr{D} = \{P_1,\dots,P_N\} \subset \Pi$, $P \in \Pi$, and $[W_{P_i}(f) \geq W_{P_i}(g)\ \forall i = 1,\dots,N] \implies W_P(f) \geq W_P(g)$ for all $f,g \in H$. Denote by $\mathbf{W}(f) = (W_{P_1}(f),\dots,W_{P_N}(f))$, $\mathbb{Q} = \mathbf{W}(H) \subset \mathbb{R}^N$, $cone(\mathbb{Q}) = \{t \in \mathbb{R}^N \,|\, \exists \tilde{t} \in \mathbb{Q} \text{ and } \gamma \geq 0 :\ t = \gamma \tilde{t}\}$ and define functions $\phi : cone(\mathbb{Q}) \to \mathbb{R}$ and $\psi : \mathbb{R}^N \to \mathbb{R}$ by*

$$\phi(t) = \gamma W_P(f) \text{ for some } \gamma > 0, f \in H \text{ such that } \mathbf{W}(f) = \gamma^{-1} t$$
$$\psi(t) = \inf\{\phi(t') \,|\, t' \in cone(\mathbb{Q}) \text{ and } t' \geq t\}$$

where $t' \geq t$ denotes $t'_i \geq t_i$ for $i = 1,\dots,N$. Then:
(a.i) Functions ϕ and ψ are well-defined;
(a.ii) $W_P(f) = \psi(\mathbf{W}(f))$ for all $f \in \mathbb{R}^S$;
(b.i) Function ψ is positively homogeneous: $\psi(\lambda t) = \lambda \psi(t)$ for all $\lambda \geq 0$;
(b.ii) Function ψ is monotone: $t \geq t' \implies \psi(t) \geq \psi(t')$;
(b.iii) Function ψ is C-additive: $\psi(t + \beta \cdot (1,\dots,1)) = \psi(t) + \psi(\beta \cdot (1,\dots,1))$ for all $\beta \in \mathbb{R}$;
(b.iv) Function ψ is normalized: $\psi(1,\dots,1) = 1$.

Proof. Denote by $e = (1,\dots,1) \in \mathbb{R}^N$, $\iota = (1,\dots,1) \in \mathbb{R}^S$.

Let $t \in cone(\mathbb{Q})$ then $\exists \gamma > 0$, $\hat{t} \in \mathbb{Q}$, and $f \in H$ such that $t = \gamma \hat{t}$ and $\mathbf{W}(f) = \hat{t} = \gamma^{-1} t$. Suppose $g \in H$, $\delta > 0$ are such that $\mathbf{W}(g) = \delta^{-1} t$. WLOG, $\delta \leq \gamma$, hence $(\delta/\gamma) g \in H$, and by positive homogeneity of $\mathbf{W}$, $\mathbf{W}((\delta/\gamma)g) = \gamma^{-1} t = \mathbf{W}(f)$. Since $\mathbf{W}(f) \geq \mathbf{W}((\delta/\gamma)g)$ and $\mathbf{W}((\delta/\gamma)g) \geq \mathbf{W}(f)$ (according to the partial order in $\mathbb{R}^N$), then $W_P(f) \geq W_P((\delta/\gamma)g)$ and $W_P((\delta/\gamma)g) \geq W_P(f)$; therefore, $W_P((\delta/\gamma)g) = W_P(f)$. Hence, $\phi(t) = \delta W_P(g) = \delta \cdot (\gamma/\delta) W_P(f) = \gamma W_P(f)$. Therefore, the value of ϕ is the same for arbitrary pairs $\gamma > 0, f \in H$ and $\delta > 0, g \in H$, proving that ϕ is well-defined. Since $\mathbf{W}(\iota) = e$, $\mathbf{W}(-\iota) = -e$, then $\alpha e \in cone(\mathbb{Q})$ for all $\alpha \in \mathbb{R}$. Since $(\min_i t_i) \cdot e \leq t \leq (\max_i t_i) \cdot e$, then ψ is well-defined as well, proving (a.i).

Claim 8. *Function ϕ is positively homogeneous, monotone, C-additive and normalized.*

Proof of Claim 8. Consider $\delta \in [0,1]$; let $f \in H$, $\gamma > 0$ be such that $\mathbf{W}(f) = \gamma^{-1} t$ for $t \in cone(\mathbb{Q})$. Then $\delta f \in H$, and $\mathbf{W}(\delta f) = \delta \mathbf{W}(f) = \gamma^{-1} \delta t$, hence $\phi(\delta t) = \gamma W_P(\delta f) = \gamma \delta W_P(f) = \delta \phi(t)$; this suffices to show positive homogeneity of ϕ.

Let $r, t \in \mathbb{Q}$ and $r \geq t$. Then $\exists f, g \in H$: $\mathbf{W}(f) = r$, $\mathbf{W}(g) = t$. Since $r \geq t$, $\phi(r) = 1 \cdot W_P(f) \geq 1 \cdot W_P(g) = \phi(t)$. Hence, ϕ is monotone on $\mathbb{Q}$. Let $r, t \in cone(\mathbb{Q})$. Since $u \in \mathbb{Q} \implies \beta u \in \mathbb{Q}\ \forall \beta \in [0,1]$ (follows from positive homogeneity of $\mathbf{W}(\cdot)$ and $H = [-1,1]^S$), then $\exists \delta > 0$ such that $\delta r, \delta t \in \mathbb{Q}$. Since ϕ is positively homogeneous, $\phi(r) = \delta^{-1} \phi(\delta r) \geq \delta^{-1} \phi(\delta t) = \phi(t)$, hence ϕ is monotone on $cone(\mathbb{Q})$.

Since $\mathbf{W}(\iota) = e$ then $\phi(e) = 1 \cdot W_P(\iota) = 1$, hence ϕ is normalized.

Let $t, t + \beta e \in cone(\mathbb{Q})$, where $\beta \in \mathbb{R}$. Then $\exists \gamma > 0$, $f \in H$: $\mathbf{W}(f) = \gamma t$. For small enough $\delta > 0$, $\delta f + \delta \gamma^{-1} \beta \iota \in H$, then $\mathbf{W}(\delta f + \delta \gamma^{-1} \beta \iota) = \delta \mathbf{W}(f) + \delta \gamma^{-1} \beta e = \gamma^{-1}(\delta t + \delta \beta e)$ and $\phi(\delta t + \delta \beta e) = \gamma W_P(\delta f + \delta \gamma^{-1} \beta \iota) = \delta \gamma W_P(f) + \delta \beta = \delta(\phi(t) + \beta) = \delta(\phi(t) + \phi(\beta e))$. Since ϕ is positively homogeneous, $\phi(t + \beta e) = \phi(t) + \phi(\beta e)$, hence ϕ is C-additive, proving Claim 8. $\square$

Let $\gamma > 0$, $t \in \mathbb{R}^S$. Using Claim 8, $\psi(\gamma t) = \inf\{\phi(t') | t' \in cone(\mathbb{Q}) \text{ and } t' \geq \gamma t\} = \inf\{\phi(t') | t' \in cone(\mathbb{Q}) \text{ and } \gamma^{-1} t' \geq t\} = \inf\{\phi(t') | \gamma^{-1} t' \in cone(\mathbb{Q}) \text{ and } \gamma^{-1} t' \geq t\} = \inf\{\phi(\gamma t'') | t'' \in cone(\mathbb{Q}) \text{ and } t'' \geq t\} = \inf\{\gamma \phi(t'') | t'' \in cone(\mathbb{Q}) \text{ and } t'' \geq t\} = \gamma \cdot \inf\{\phi(t'') | t'' \in cone(\mathbb{Q}) \text{ and } t'' \geq t\} = \gamma \psi(t)$. Thus, ψ is positive homogeneous, proving (b.i).

Let $t' \geq t$, then $\{t'' \in cone(\mathbb{Q}) | t'' \geq t'\} \subseteq \{t'' \in cone(\mathbb{Q}) | t'' \geq t\}$, hence $\psi(t') \geq \psi(t)$, proving (b.ii).

Claim 9. *For all $t \in cone(\mathbb{Q})$, $\psi(t) = \phi(t)$.*

Proof of Claim 9. $[t' \geq t \implies \phi(t') \geq \phi(t)] \implies \psi(t) \geq \phi(t)$; $t \in cone(\mathbb{Q}) \implies \psi(t) \leq \phi(t)$. $\square$

Using Claims 8,9, $\psi(e) = \phi(e) = 1$, proving (b.iv).

Claim 10. *If $t \in cone(\mathbb{Q})$, then $t + \beta e \in cone(\mathbb{Q})$ for all $\beta \in \mathbb{R}$.*

Proof of Claim 10. It is enough to consider $\beta \neq 0$. Since $t \in cone(\mathbb{Q})$, $\exists \gamma > 0$, $f \in H$: $t = \gamma \mathbf{W}(f)$. For small enough $\delta > 0$, $\delta f + \delta \gamma^{-1} \beta \iota \in H$, hence $t + \beta e = \delta^{-1} \gamma \mathbf{W}(\delta f + \delta \gamma^{-1} \beta \iota) \in cone(\mathbb{Q})$. $\square$

Using Claim 10, we get $\psi(t+\beta e) = \inf\{\phi(t')\big|t' \in cone(\mathbb{Q})$ and $t' \geq t+\beta e\} = \inf\{\phi(t')\big|t' \in cone(\mathbb{Q})$ and $t' - \beta e \geq t\} = \inf\{\phi(t')\big|t' - \beta e \in cone(\mathbb{Q})$ and $t' - \beta e \geq t\} = \inf\{\phi(t'' + \beta e)\big|t'' \in cone(\mathbb{Q})$ and $t'' \geq t\} = \inf\{\phi(t'') + \beta\big|t'' \in cone(\mathbb{Q})$ and $t'' \geq t\} = \inf\{\phi(t'')\big|t'' \in cone(\boldsymbol{Q})$ and $t'' \geq t\} + \beta = \psi(t) + \beta = \psi(t) + \beta\psi(e) = \psi(t) + \psi(\beta e)$, proving (b.iii).

Let $f \in \mathbb{R}^S$, then $\delta f \in H$ for some $\delta > 0$. Using Claim 9 and (b.i), we get $W_P(f) = \delta^{-1}W_P(\delta f) = \delta^{-1}\phi(\mathbf{W}(\delta f)) = \delta^{-1}\psi(\mathbf{W}(\delta f)) = \delta^{-1}\psi(\delta\mathbf{W}(f)) = \psi(\mathbf{W}(f))$, proving (a.ii). Lemma 14 is proven. □

We next use results from Chandrasekher et al. (2022).

Lemma 15. *Let function $\psi: \mathbb{R}^N \to \mathbb{R}$ be positively homogeneous, monotone, C-additive and normalized, then there is a non-empty compact collection Θ of non-empty compact and convex sets of weights $\Lambda \subseteq \triangle(\{1,...,N\})$ such that for all $t \in \mathbb{R}^N$*

$$\psi(t) \;=\; \max_{\Lambda\in\Theta}\min_{\lambda\in\Lambda}\sum_{i=1}^{N}\lambda_i t_i \tag{8}$$

Proof. The statement of this Lemma is equation 19 in Appendix B1 on page 29 of Chandrasekher et al. (2022). □

Lemma 16. *If $c(\cdot)$ has a framed ambiguity representation with finite number of frames, then any framed ambiguity representation of $c(\cdot)$ has a finite number of frames.*

Proof. Follows from Theorem 2, since $|\mathscr{A}| < \infty \implies |(\mathscr{A})_{coh}| \leq 2^{|\mathscr{A}|} < \infty$. □

Lemma 17. *If P is a convex union of $\mathscr{C}$ (eq. (3)), then $P \in \Pi$ and $I_P(\mathbf{z}) = \min_{P'\in\mathscr{C}} I_{P'}(\mathbf{z})\ \forall \mathbf{z} \in \mathbb{R}^S$.*

Proof. Note that P is non-empty and convex. Let $\mu^n \in \bigcup_{P'\in\mathscr{C}} P'$ be such that $\mu^n \longrightarrow \mu \in \Pi$. Then $\forall n = 1,2,...\ \exists P^n \in \mathscr{C}$: $\mu^n \in P^n$. Since $\mathscr{C}$ is compact, $\exists P^{n_k} \longrightarrow P' \in \mathscr{C}$. Thus, $\exists \hat{\mu}^{n_k} \in P'$ for $k = 1,2,...$ such that $|\hat{\mu}^{n_k} - \mu^{n_k}| \longrightarrow 0$. Therefore, $|\hat{\mu}^{n_k} - \mu| \leq |\hat{\mu}^{n_k} - \mu^{n_k}| + |\mu^{n_k} - \mu| \to 0$. Since P' is closed, $\hat{\mu}^{n_k} \longrightarrow \mu \in P'$. Thus, the set $\tilde{P} = \bigcup_{P'\in\mathscr{C}} P'$ is closed. Therefore, its convex hull P is closed (by Corollary 5.33 of Aliprantis and Border (2005)), and $P \in \Pi$. Finally,

$$I_P(\mathbf{z}) \;=\; \min_{\mu\in conv(\tilde{P})}\sum_{s\in S}\mu(s)\mathbf{z}_s \;=\; \min_{\mu\in\tilde{P}}\sum_{s\in S}\mu(s)\mathbf{z}_s \;=\; \min_{P'\in\mathscr{C}}\min_{\mu\in P'}\sum_{s\in S}\mu(s)\mathbf{z}_s \;=\; \min_{P'\in\mathscr{C}}\ I_{P'}(\mathbf{z})$$ □

Lemma 18. *Let $\mathscr{C} = \{P_1,...,P_N\} \subseteq \Pi$, $\lambda \in \triangle(\{1,...,N\})$. Then:*
(i) If P is a convex combination of $\mathscr{C}$ with respect to λ (eq. (4)), then $P \in \Pi$ and $I_P(\mathbf{z}) = \sum_{i=1}^{N}\lambda_i I_{P_i}(\mathbf{z})$;
(ii) Function $\zeta: \triangle(\{1,...,N\}) \to \Pi$ given by $\zeta(\lambda) = \sum_{i=1}^{N}\lambda_i P_i$ is continuous.

Proof. Note that P is non-empty. Let $\mu^n \in P$ for $n = 1,2,...$, and $\mu^n \longrightarrow \mu$. Then $\exists \mu_i^n : \mu_i^n \in P_i$

$\forall n=1,2,\ldots\ \forall i=1,\ldots,N$ and $\mu^n=\sum_{i=1}^N \lambda_i \mu_i^n\ \forall n=1,2,\ldots$. Since $N<\infty$, $\exists \mu^{n_k}$, $\mu_i\in P_i$, $i=1,\ldots,N$: $\mu_i^{n_k} \longrightarrow \mu_i\ \forall i=1,\ldots,N$. Then $\mu=\lim\limits_{k\to\infty}\mu^{n_k}=\lim\limits_{k\to\infty}\sum_{i=1}^N \lambda_i\mu_i^{n_k}=\sum_{i=1}^N\lambda_i\mu_i$, hence $\mu\in P$, and P is closed. If $\mu,\mu'\in P$, then there are $\mu_i,\mu_i'\in P_i$ for $i=1,\ldots,N$ such that $\mu=\sum_{i=1}^N\lambda_i\mu_i$ and $\mu'=\sum_{i=1}^N\lambda_i\mu_i'$. Since each P_i is convex, $\rho\mu_i+(1-\rho)\mu_i'\in P_i$ for all $\rho\in(0,1)$, hence $\rho\mu+(1-\rho)\mu'=\sum_{i=1}^N\lambda_i(\rho\mu_i+(1-\rho)\mu_i')$, and P is convex. Finally, to prove statement (i),

$$I_P(\mathbf{z})=\min_{\mu\in P}\sum_{s\in S}\mu(s)z_s=\min_{\mu_i\in P_i\ \forall i=1,\ldots,N}\sum_{s\in S}\sum_{i=1}^N\lambda_i\mu_i(s)z_s=\sum_{i=1}^N\lambda_i\left(\min_{\mu_i\in P_i}\sum_{s\in S}\mu_i(s)z_s\right)=\sum_{i=1}^N\lambda_i I_{P_i}(\mathbf{z})$$

Let $\mu\in\sum_{i=1}^N\lambda_iP_i$, then $\exists\mu_i\in P_i$: $\mu=\sum_{i=1}^N\lambda_i\mu_i$. Consider $\mu'=\sum_{i=1}^N\lambda_i'\mu_i\in\sum_{i=1}^N\lambda_i'P_i$. Then $\mu-\mu'=\sum_{i=1}^N(\lambda_i-\lambda_i')\mu_i\longrightarrow 0$ when $\lambda'\longrightarrow\lambda$. Similarly, $\forall\mu'\in\sum_{i=1}^N\lambda_i'P_i\ \exists\mu\in\sum_{i=1}^N\lambda_iP_i$: $\mu'-\mu\longrightarrow 0$ when $\lambda'\longrightarrow\lambda$. Therefore, $\lambda'\longrightarrow\lambda$ implies $\sum_{i=1}^N\lambda_i'P_i\longrightarrow\sum_{i=1}^N\lambda_iP_i$, proving (ii). □

The next lemma mirrors Proposition 1 in Crès et al. (2011):

Lemma 19. *Let $\Lambda\subseteq\triangle(1,\ldots,N)$ be a non-empty closed and convex set of weights, then*

$$W_P(\mathbf{z})\ =\ \min_{\lambda\in\Lambda}\sum_{i=1}^N\lambda_iW_{P_i}(\mathbf{z})\ \ \forall\mathbf{z}\in\mathbb{R}^S\quad \textit{iff}\quad P\ =\ \left\{\mu\in\triangle S\ \middle|\ \exists\lambda\in\Lambda \textit{ and } \mu_i\in P_i:\ \mu=\sum_{i=1}^N\lambda_i\mu_i\right\}\tag{9}$$

Proof. Let $P=\left\{\mu\in\triangle S\ \middle|\ \exists\lambda\in\Lambda\text{ and }\mu_i\in P_i:\ \mu=\sum_{i=1}^N\lambda_i\mu_i\right\}$. For an arbitrary $\mathbf{z}\in\mathbb{R}^S$ let $\mu_i^*\in\arg\min\limits_{\mu\in P_i}(\mu\cdot\mathbf{z})\neq\varnothing$ and $\lambda^*=\arg\min\limits_{\lambda\in\Lambda}\sum_{i=1}^N\lambda_iW_{P_i}(\mathbf{z})\neq\varnothing$; the latter minimum exists because by Lemma 18 (ii), the convex combination of sets of beliefs is continuous in weights λ, and Λ is compact. Then $\mu^*=\sum_{i=1}^N\lambda_i^*\mu_i^*\in P$, and $W_P(\mathbf{z})\leq\mu^*\cdot\mathbf{z}=\sum_{i=1}^N\lambda_i^*\mu_i^*\cdot\mathbf{z}=\min\limits_{\lambda\in\Lambda}\sum_{i=1}^N\lambda_iW_{P_i}(\mathbf{z})$.

Conversely, let $\mu^{**}\in\arg\min\limits_{\mu\in P}\mu\cdot\mathbf{z}$. Then $\exists\lambda^{**}\in\Lambda$, $\mu_i^{**}\in P_i$: $\mu=\sum_{i=1}^N\lambda^{**}\mu_i^*$, and $W_P(\mathbf{z})=\mu^{**}\cdot\mathbf{z}=\sum_{i=1}^N\lambda_i^{**}\mu_i^{**}\cdot\mathbf{z}\geq\sum_{i=1}^N\lambda_i^{**}W_{P_i}(\mathbf{z})\geq\min\limits_{\lambda\in\Lambda}\sum_{i=1}^N\lambda_iW_{P_i}(\mathbf{z})$. Hence, $W_P(\mathbf{z})\ =\ \min\limits_{\lambda\in\Lambda}\sum_{i=1}^N\lambda_iW_{P_i}(\mathbf{z})$, proving the "if" direction of the Lemma. The "only if" direction follows from Lemma 8. □

Lemma 20. *(i) For any non-empty collection of sets of beliefs $\mathscr{A}\subseteq\Pi$ its closure $\Gamma(\mathscr{A})$ with respect to operations of convex union, coherent intersection and convex combination given by Definition 8 exists and unique; (ii) for any any non-empty collection of sets of beliefs $\mathscr{A},\mathscr{B}$: (ii.a) $\mathscr{A}\subseteq\mathscr{B}\implies\Gamma(\mathscr{A})\subseteq\Gamma(\mathscr{B})$; (ii.b) $\Gamma(\Gamma(\mathscr{A}))=\Gamma(\mathscr{A})$.*

Proof. Let $\mathbb{F}$ be the set of collections of sets of beliefs $\mathscr{F}$ that satisfy condition (i) and (ii) of the Definition 8. Note that $\Pi\in\mathbb{F}\neq\varnothing$. Then there exists $\Gamma(\mathscr{A})=\bigcap\limits_{\mathscr{F}\in\mathbb{F}}\mathscr{F}$ and it is unique, proving (i). Next, if $\mathscr{A}\subseteq\mathscr{B}$, then $\Gamma(\mathscr{B})$ is closed under the three considered operations, and it contains $\mathscr{A}\subseteq\mathscr{B}$, hence $\Gamma(\mathscr{A})\subseteq\Gamma(\mathscr{B})$, proving (ii.a). Since $\Gamma(\mathscr{A})$ is closed under the three considered operations and it contains itself, then $\Gamma(\Gamma(\mathscr{A})) = \Gamma(\mathscr{A})$, proving (ii.b). $\square$

Lemma 21. *Let vNM expected utility function $U(\cdot)$ be fixed, and $\varnothing \neq \mathscr{D} \subseteq \Pi$. If $P \in \Gamma(\mathscr{D})$, then $[f \succ_{P'} g \;\; \forall P' \in \mathscr{D}]$ implies $f \succ_P g$.*

Proof. Let $\mathscr{E} \subseteq \Gamma(\mathscr{D})$ be the collection of all sets of beliefs $P \in \Gamma(\mathscr{D})$ such that $\left[f \succ_{P'} g \; \forall P' \in \mathscr{D}\right]$ implies $f \succ_P g$. Note that $\varnothing \neq \mathscr{D} \subseteq \mathscr{E}$. Let f and g be such that $f \succ_{P'} g \; \forall P' \in \mathscr{D}$. Take arbitrary closed sub-collection $\mathscr{C} \subseteq \mathscr{E}$. If it intersects coherently, and its intersection is P, by Lemma 11, $W_P(f) = \max_{Q \in \mathscr{C}} W_Q(f) \geq W_{Q^*}(f) > W_{Q^*}(g) = \max_{Q \in \mathscr{C}} W_Q(g) = W_P(g)$, where Q^* maximizes $W_Q(g)$ over $\mathscr{C} \subseteq \mathscr{E}$ (exists since $\mathscr{C}$ is compact), and $W_{Q^*}(f) > W_{Q^*}(g)$ by the definition of $\mathscr{E}$; hence, $f \succ_P g$ and $P \in \mathscr{E}$.

If P is a convex union of $\mathscr{C}$, then by Lemma 17, $W_P(f) = \min_{Q \in \mathscr{C}} W_Q(f) = W_{Q^{**}}(f) > W_{Q^{**}}(g) \geq \min_{Q \in \mathscr{C}} W_Q(g) = W_P(g)$, where Q^{**} minimizes $W_Q(f)$ over $\mathscr{C} \subseteq \mathscr{E}$; hence, $f \succ_P g$ and $P \in \mathscr{E}$. Finally, if $\mathscr{C} = \{Q_1, \ldots, Q_N\}$ is finite, and P is a convex combination of frames in $\mathscr{C}$ with weights λ, then by Lemma 18, $W_P(f) = \sum_{i=1}^N \lambda_i W_{Q_i}(f) > \sum_{i=1}^N \lambda_i W_{Q_i}(g) = W_P(g)$, where we used $W_{Q_i}(f) > W_{Q_i}(g)$ since $Q_i \in \mathscr{C} \subseteq \mathscr{E}$. Again, $f \succ g$ and $P \in \mathscr{E}$. Therefore, $\Gamma(\mathscr{E}) = \mathscr{E}$, and $\Gamma(\mathscr{D}) \subseteq \Gamma(\mathscr{E}) \subseteq \Gamma(\Gamma(\mathscr{D})) = \Gamma(\mathscr{D})$, which implies $\mathscr{E} = \Gamma(\mathscr{D})$, proving the Lemma. $\square$

We now prove the statements of the Theorem. Suppose first that DM 1 is more consistent than DM 2. By Lemma 13 ((i) $\Longrightarrow$ (iv)), U_1 is a positive affine transformation of U_2, and $[f \succeq_P g \; \forall P \in \mathscr{A}_2] \Longrightarrow [f \succeq_P g \; \forall P \in \mathscr{A}_1]$. By Lemma 5, it is WLOG to consider $X = \{x, y\}$. Normalize $U(x) = 1, U(y) = -1$, and consider $P \in \mathscr{A}_1$, then by Lemma 14, $W_P(f) = \psi(\mathbf{W}(f))$ for all f, where ψ is positively homogeneous, monotone, C-additive and normalized. Therefore, by Lemma 15, there is a non-empty compact collection Θ of non-empty compact and convex sets of weights $\Lambda \subseteq \Delta(\{1, \ldots, N\})$ such that for all $t \in \mathbb{R}^N$, $\psi(t) = \max\limits_{\Lambda \in \Theta} \min\limits_{\lambda \in \Lambda} \sum\limits_{i=1}^N \lambda_i t_i$. Hence, for all $f \in \mathbb{R}^S$,

$$W_P(f) \;=\; \max_{\Lambda \in \Theta}\left(\min_{\lambda \in \Lambda}\left(\sum_{i=1}^N \lambda_i W_{P_i}(f) \right)\right) \;=\; \max_{\Lambda \in \Theta} W_{P_\Lambda}(f)$$

where $P_\Lambda = \left\{ \mu \in \Delta S \;\middle|\; \exists \lambda \in \Lambda \text{ and } \mu_i \in P_i : \; \mu = \sum_{i=1}^N \lambda_i \mu_i \right\}$ and we used Lemma 19. Therefore, by Lemma 11, Θ is compact, and P is a coherent intersection of the collection of sets of beliefs $\{P_\Lambda\}_{\Lambda \in \Theta}$. By Lemmas 17, 18, each P_Λ is a convex union of the family $\{P_\lambda\}_{\lambda \in \Lambda}$, and each P_λ is a convex combination of P_i with weights λ. Hence, statement (i) of the Theorem implies (iii).

The implication (iii) $\Longrightarrow$ (ii) is straightforward: each of the three operations results in a set of beliefs in $\Gamma(\mathscr{A}_2)$ by the definition of $\Gamma(\cdot)$. Finally, assume statement (ii) holds. By Lemma 21,

$[f \succ_{P'} g \;\; \forall P' \in \mathscr{D}]$ implies $f \succ_P g$ for all $P \in \mathscr{A}_1 \subseteq \Gamma(\mathscr{A}_2)$. Therefore, by Lemma 13 ((v) $\Longrightarrow$ (i)), DM 1 is more consistent than DM 2. Thus, $(i) \Longrightarrow (iii) \Longrightarrow (ii) \Longrightarrow (i)$, proving the Theorem. ■

Proof of Proposition 4

Proposition 4 is proven in Lemma 13 ((i)$\Longleftrightarrow$(iii)). ■

Proof of Corollary 2

Suppose $\succeq_1 = \succeq_2$; that is, $c_1(\{f,g\}) = c_2(\{f,g\})$ for all $f,g \in H$. By Lemma 13 ((iii) $\Longrightarrow$ (i)), DM 1 is more consistent than DM 2, and vice versa, DM 2 is more consistent than DM 1. By Theorem 3, U_2 is a positive affine transformation of U_1, $\mathscr{A}_1 \subseteq \Gamma(\mathscr{A}_2)$, $\mathscr{A}_2 \subseteq \Gamma(\mathscr{A}_1)$, hence $\Gamma(\mathscr{A}_2) = \Gamma(\mathscr{A}_1)$.

Suppose U_2 is a positive affine transformation of U_1 and $\Gamma(\mathscr{A}_2) = \Gamma(\mathscr{A}_1)$. By Theorem 3, DM 1 is more consistent than DM 2, and vice versa, DM 2 is more consistent than DM 1. By Lemma 13 ((i) $\Longrightarrow$ (iii)), $c_1(\{f,g\}) = c_2(\{f,g\})$ for all $f,g \in H$, hence $\succ_1 = \succeq_2$. ■

Proof of Corollary 3

Consider DM 2 whose choices are represented by the framed ambiguity model $(U, \{P_i\}_{i=1,\dots,N})$, where $U(\cdot)$ is a common vNM expected utility function, and DM 1 with choice correspondence $c_1(\cdot)$ represented by $(U, \{P\})$. These models are well-defined, since $U(\cdot)$ is non-degenerate because of non-degeneracy of $\succeq_i$, $\succeq$, and families of frames are finite and, hence, closed. Therefore, by Lemma 13 ((i)$\Longleftrightarrow$(iv)), $\succeq$ satisfies *Unanimity* with respect to $(U, \{P_i\}_{i=1,\dots,N})$ if and only if DM 1 is more consistent than DM 2, which is equivalent to $\{P\} \subseteq \Gamma(\{P_1, \dots, P_N\})$ by Theorem 3. ■

Proof of Proposition 5

Suppose (ii) holds, then without loss, $U_1 = U_2$. Suppose $f_i \in c_2(\{f_i, p\})$ $\forall i = 1, \dots, k$, then

$$W_P\Big(\sum_{i=1}^{k} \sigma_i f_i\Big) \;\geq\; \sum_{i=1}^{k} \sigma_i W_P(f_i) \;\geq\; \sum_{i=1}^{k} \sigma_i \cdot \max_{Q \in \mathscr{A}_2} W_Q(f_i) \;\geq\; \sum_{i=1}^{k} \sigma_i \cdot U(p) \;=\; U(p)$$

where we used the concavity of the maxmin expected utility, and the fact that $\max\limits_{Q \in \mathscr{A}_2} W_Q(f_i) \leq \mu \cdot U(f(i))$ for all $\mu \in P \subseteq \bigcap\limits_{Q \in \mathscr{A}_2} Q$. Hence, $\sum_{i=1}^{k} \sigma_i f_i \in c_1\Big(\Big\{\sum_{i=1}^{k} \sigma_i f_i,\; p\Big\}\Big)$, proving (i).

Lemma 22. *If statement (i) of Proposition 5 holds, then U_1 is a positive affine transformation of U_2.*

Proof. Note that $U_2(p) \geq U_2(q) \Longleftrightarrow p \in c_2(\{p,q\}) \implies p \in c_1(\{p,q\}) \implies U_1(p) \geq U_1(q)$. Since U_1 is non-degenerate, $\exists x, y \in X$: $U_1(x) > U_1(y)$; this implies $U_2(x) > U_2(y)$. Normalize both U_1

and U_2 such that $U_1(x) = U_2(x) = 1$, $U_1(y) = U_2(y) = 0$. If $U_2(p) = \lambda^{-1} > 1$, then $U_2(\lambda p + (1-\lambda)y) = U_2(x) = 1 \implies U_1(\lambda p + (1-\lambda)y) = U_2(x) = 1$. Hence, $U_1(p) = \lambda^{-1} = U_2(p)$. Similar analysis for $U_2(p) \in [0,1]$ and $U_2(p) < 0$ shows that $U_1(p) = U_2(p)$ for all $p \in \triangle X$. $\square$

Lemma 23. *If $\mathscr{A}$ is a non-empty compact family of non-empty compact sets of beliefs, then $\bigcap_{Q \in \mathscr{A}} Q \neq \varnothing$ if and only if $\bigcap_{Q \in \mathscr{C}} Q \neq \varnothing \quad \forall \mathscr{C} \subseteq \mathscr{A}: 1 \leq |\mathscr{C}| < \infty$.*

Proof. Since $\triangle S$ has the Heine-Borel property, if $\mathscr{A}$ has a finite intersection property, then $\bigcap_{Q \in \mathscr{A}} Q \neq \varnothing$. The other direction is trivial. The Online Appendix provides a direct proof. $\square$

Let statement (i) of Proposition 5 holds; by Lemma 22, WLOG, $U_1 = U_2$. Towards a contradiction, assume $\exists \mu \in \left(P_1 \setminus \bigcap_{Q \in \mathscr{A}_2} Q\right) \neq \varnothing$. Since $\mathscr{A} = \mathscr{A}_2 \cup \{\{\mu\}\}$ is compact, and $\{\mu\} \cap \left(\bigcap_{Q \in \mathscr{A}_2} Q\right) = \varnothing$, then by Lemma 23, there is a finite sub-family $\{K_1, \ldots, K_N\} \subset \mathscr{A}$ such that $\bigcap_{i=1}^{N} K_i = \varnothing$; we may assume that $K_1 = \{\mu\}$, and $K_2, \ldots, K_N \in \mathscr{A}_2$ without loss. We next use a proposition from Samet (1998):

Lemma 24. *Let $K_1, \ldots, K_N \in \Pi$, then $\bigcap_{i=1}^{N} K_i = \varnothing$ if and only if $\exists z_1, \ldots, z_N \in \mathbb{R}^S$: $\sum_{i=1}^{N} z_i = 0$, and $z_i \cdot \mu_i > 0 \ \forall \mu_i \in K_i \forall i = 1, \ldots, N$.*

Proof. See Samet (1998) $\square$

By Lemma 24, $\exists z_1, \ldots, z_N \in \mathbb{R}^S$: $-z_1 = \sum_{i=2}^{N} z_i$, $-z_1 \cdot \mu < 0$, and $\min_{\nu \in K_i}(\nu \cdot z_i) \geq 0$ for all $i = 2, \ldots, N$. Dividing all z_i by the same positive number does not change the conclusion, hence $|z_i| \leq 0.1$ without loss. WLOG, $U_1(x) = U_2(x) = 1$, $U_1(y) = U_2(y) = 0$ for some $x, y \in X$. Consider acts f_i, $i = 2, \ldots, N$ given by $f_i(s) = \big(0.5 + (z_i)_s\big)x + \big(0.5 - (z_i)_s\big)y$, and note that the act $g = \sum_{i=2}^{N} \frac{1}{N-1} f_i$ is given by $g(s) = \big(0.5 - (z_1)_s/(N-1)\big)x + \big(0.5 + (z_1)_s/(N-1)\big)y$. Therefore, $W_{K_i}(f_i) \geq 0.5 = U_2(0.5x + 0.5y)$ for all $i = 2, \ldots, N$, and $W_P(g) \leq 0.5 + (-z_1 \cdot \mu)/(N-1) < 0.5 = U_1(0.5x + 0.5y)$. It follows $f_i \in c_2(\{f_i, 0.5x + 0.5y\})$ for all $i = 2, \ldots, N$, but $g \notin c_1(\{g, 0.5x + 0.5y\})$. This contradiction proves the implication (i) $\implies$ (ii). $\blacksquare$

Proof of Proposition 6

Lemma 25. *Let Condition 1 or 2 hold. Then*

$$I_{\bigcap_{i=1,\ldots,N} P_i}(\mathbf{z}) \quad = \quad sup\left\{ \sum_{i=1}^{N} I_{P_i}(\mathbf{z}_i) \;\middle|\; \sum_{i=1}^{N} \mathbf{z}_i = \mathbf{z} \right\} \tag{10}$$

where $I_P(\mathbf{z}) = \min_{\mu\in P}(\mu\cdot\mathbf{z})$, and for each $\mathbf{z}\in\mathbb{R}^S$, the suprenum is attained[27].

Proof. If Condition 2 holds, the statement follows from Corollary 16.4.1 in Rockafellar (1970). If Condition 1 holds, the statement follows from Theorem 20.1 in Rockafellar (1970) for the indicator functions $f_i(\cdot) = \delta(\cdot|P_i)$, where $f_i(\cdot)$ are polyhedral because P_i are polyhedral (Corollary 19.2.1 in Rockafellar (1970)). ☐

Denote by $\{Q_1,...,Q_N\} = \mathscr{A}$ and $P = \bigcap_{i=1}^{N} Q_i$. Normalize the expected utility such that $U(x) = 1$, $U(y) = 0$ for some $x, y \in X$, and let $q = 0.5x + 0.5y$. By Lemma 25, $\exists z_1,...,z_N \in \mathbb{R}^S$ such that $\sum_{i=1}^{N}(z_i)_s = U(f(s))$ for $s\in S$, and $W_P(f) = I_P(U(f)) = \sum_{i=1}^{N} I_{P_i}(z_i)$. Next, let $\lambda \in (0,1]$, and consider, for $i = 1,...,N$, the following acts f_i that are well-defined for sufficiently small λ:

$$f_i(s) = \lambda f(s) + \Big[(1-\lambda)0.5 - \lambda U(f(s)) + \lambda N(z_i)_s + \lambda\big(W_P(f) - N\cdot I_{Q_i}(z_i)\big)\Big]x+$$
$$+\Big[(1-\lambda)0.5 + \lambda U(f(s)) - \lambda N(z_i)_s - \lambda\big(W_P(f) - N\cdot I_{Q_i}(z_i)\big)\Big]y$$

By our choice of z_i, we have $\sum_{i=1}^{N}\sigma_i f_i = \lambda f + (1-\lambda)(0.5x+0.5y) = \lambda f + (1-\lambda)q$, where $\sigma_i = N^{-1}$. Next, $U(f_i(s)) = (1-\lambda)0.5 + \lambda N(z_i)_s + \lambda\big(W_P(f) - N\cdot I_{Q_i}(z_i)\big)$, hence $I_{Q_i}(U(f_i)) = (1-\lambda)0.5 + \lambda W_P(f) = W_P(\lambda f + (1-\lambda)q)$ for all $i = 1,...,N$. Therefore, $\lambda f + (1-\lambda)q \in c_1(\{\lambda f + (1-\lambda)q, p\}) \implies W_P(\lambda f + (1-\lambda)q) \geq U(p) \implies I_{Q_i}(U(f_i)) \geq U(p) \implies f_i \in c_2(\{f_i, p\})$ for all $i = 1,...,N$. From the other hand, by Proposition 5, $f_i \in c_2(\{f_i, p\})$ for all $i = 1,...,N$ implies $\lambda f + (1-\lambda)q \in c_1(\{\lambda f + (1-\lambda)q, p\})$. Thus, we have proven that statement (ii) of Proposition 6 implies statement (i).

Lemma 26. *If c_1 satisfies statement (i) of Proposition 6, then it is unique and it is given by $c_1(A) = \{f\in A | f \succeq g\ \forall g\in A\}$, where $f \succeq g$ if and only if $\exists p\in H_0$: $f\in c_1(\{f,p\})$ and $p\in c_1(\{g,p\})$.*

Claim 11. *If c_1 satisfies statement (i) of Proposition 6, then $c_1(\{r,p\}) = c_2(\{r,p\})$ for all $r, p \in H_0$*

Proof of Claim 11. Consider a decomposition $r = f_1$; then, $r\in c_2(\{r,p\}) \implies r\in c_1(\{r,p\})$. Next, by Condition 1 or Condition 2, $\exists\mu\in\bigcap_{Q\in\mathscr{A}} Q \neq \varnothing$. Let $\lambda r + (1-\lambda)q = \sum_{i=1}^{k}\sigma_i f_i$ be an arbitrary decomposition with $\lambda\in(0,1]$. Then

$$\sum_{i=1}^{k}\sigma_i \max_{Q\in\mathscr{A}} W_Q(f_i) \leq \sum_{i=1}^{k}\sigma_i\sum_{s\in S}\mu_s U(f_i(s)) = \sum_{s\in S}\mu_s U\big(\sum_{i=1}^{k}\sigma_i f_i(s)\big) = \lambda U(r) + (1-\lambda)U(q)$$

If $r\in c_1(\{r,p\})$, then for some decomposition, $\max_{Q\in\mathscr{A}} W_Q(f_i) \geq \lambda U(p) + (1-\lambda)U(q), i = 1,...,k$, It follows that $U(r) \geq U(p)$ and $r\in c_2(\{r,p\})$. ☐

Claim 12. *Let c_1 satisfies statement (i) of Proposition 6, then there is $p_f\in H_0$ such that* [$p\in$

[27] We define the support functional $I_P(\cdot)$ as the minimum of a linear function over P, while Rockafellar (1970) defines it as a maximum. Hence, we get sup instead of his inf in lemma 25.

$c_1(\{f,p\})$ *if and only if* $U(p) \geq U(p_f)$], *and* [$f \in c_1(\{f,p\})$ *if and only if* $U(p) \leq U(p_f)$].

Proof of Claim 12. Let $\overline{p}$, $\underline{p} \in \triangle X$ be such that $U(\underline{p}) = \min_{s\in S} U(f(s))$, $U(\overline{p}) = \min_{s\in S} U(f(s))$; one can take $\overline{p} = f(s')$, $\underline{p} = f(s'')$ for s', s'' chosen accordingly. If $U(\underline{p}) = U(\overline{p})$, we are done by Claim 11, so, consider $U(\underline{p}) < U(\overline{p})$. Consider a decomposition $f = f_1$, then $f \in c_2(\{f, \underline{p}\})$, and hence, $f \in c_1(\{f, \underline{p}\})$. Next, consider $f^n = (1-1/n)f + (1/n)\underline{p}$, and let $\lambda f^n + (1-\lambda)q = \sum_{i=1}^{k} \sigma_i f_i$ be an arbitrary decomposition. Let $\mu \in \bigcap_{Q\in\mathscr{A}} Q \neq \varnothing$. Assume $f_i \in c_2(\{f_i, \lambda\overline{p} + (1-\lambda)q\})$ for $i = 1, \ldots, k$. Then $\lambda U(\overline{p}) + (1-\lambda)U(q) \leq \sum_{s\in S} \mu_s U(f_i(s))$ for $i = 1, \ldots, k$, but

$$\sum_{s\in S} \mu_s U(f_i(s)) = \sum_{i=1}^{k} \sigma_i \sum_{s\in S} \mu_s U(f_i(s)) = \sum_{s\in S} \mu_s U(\lambda f^n + (1-\lambda)q) \leq \lambda \frac{n-1}{n} U(\overline{p}) + \frac{\lambda}{n} U(\underline{p}) + (1-\lambda)U(q)$$

which implies $U(\overline{p}) \leq U(\underline{p})$ in contradiction. Hence, $f^n \notin c_1(\{f^n, \overline{p}\})$, and $\overline{p} \in c_1(\{f^n, \overline{p}\})$. By continuity of c_1, $\overline{p} \in c_1(\{f, \overline{p}\})$. Therefore, by continuity of c_1, $\exists p_f = \zeta \underline{p} + (1-\zeta)\overline{p}$ with $\zeta \in [0,1]$ such that $c_1(\{f, p_f\}) = \{f, p_f\}$. Since c_1 satisfies *WARP*, the statement of the Claim follows. □

Proof of Lemma 26. By Claims 11, 12, and *WARP*, there is a utility function $W : H \to \mathbb{R}$ given by $W(f) = U(p_f)$ such that $f \in c(\{f,g\})$ if and only if $W(f) \geq W(g)$. By *WARP*, $c_1(A) = \{f \in A | W(f) \geq W(g) \forall g \in A\}$. Since W is identified via c_1 on menus $\{f,p\}$, $f \in H$, $p \in H_0$, the Lemma follows. □

Let $\hat{c}$ be the choice correspondence induced by the Gilboa and Schmeidler (1989) maxmin model with vNM expected utility function U and set of priors $\bigcap_{Q\in\mathscr{A}} Q$, then $\hat{c}$ satisfies statement (i) of Proposition 6 and coincides with c_1 on menus $\{f,p\}$, $f \in H$, $p \in H_0$. By Lemma 26, $\hat{c}$ coincides with c_1 for all menus, proving that statement (i) of Proposition 6 implies statement (ii). ■

Proofs of other statements

Lemma 27. *There is a choice correspondence $c(\cdot)$ that satisfies axioms 1-6 and violates Axiom 7 (C-Non-Degeneracy) such that $c(A) \neq A$ for some $A \in \mathscr{K}$; moreover, $c(\{f,g\}) \neq \{f,g\}$ for some $f,g \in H$ and $c(A) = A$ for all $A \in \mathscr{K}_0$.*

Proof. Consider $c(A) = \{f \in A \big| f(1)(x) \geq f(2)(x) \text{ or } f(2)(x) \geq g(2)(x)\ \forall g \in A\}$. Clearly, $\varnothing \neq c(A) \subseteq A$. Consider $f, g \in H$ with $f(1)(x) = 0$, $f(2)(x) = 0.5$, $g(1)(x) = 0$, $g(2)(x) = 1$. Then $c(\{f,g\}) = \{g\}$. Since $f(1)(x) = f(2)(x)$ for all $f \in H_0$, then $c(A) = A$ for all $A \in \mathscr{K}_0$, and *C-Non-Degeneracy* is violated. The proof that other axioms are satisfied is omitted. ■

Lemma 28. *If $P_\theta = \bigcap_{i_\theta \in I_\theta} P_{i_\theta}$ is a coherent intersection for all $\theta \in \Theta$, and $P = \bigcap_{\theta\in\Theta} P_\theta$ is a coherent intersection, then $P = \bigcap_{j \in \{I_\theta\}_{\theta\in\Theta}} P_{i_\theta}$ is a coherent intersection as well.*

Proof. Using Lemma 11 repeatedly, $I_P = \max_{\theta\in\Theta}\max_{i_\theta\in I_\theta} I_{P_{i_\theta}} = \max_{j\in\{I_\theta\}_{\theta\in\Theta}} I_{P_j}$. ■

Lemma 29. *Consider example given on pages 17-18. The intersection $P_3 = P_5 \cap P_6$ is coherent.*

Proof. Note that $\{\mu_1,\mu_3,\mu_4\}$ is the set of extreme points of P_5, and similarly, $P_6 = conv(\{\mu_2,\mu_3,\mu_4\})$, $P_3 = conv(\{\mu_3,\mu_4\})$. Since $0.5\mu_1+0.5\mu_2 = (0.2,0.2,0.6) = 0.5\mu_3+0.5\mu_4$, $\forall f\in H$, it is not possible that

$$\sum_{s=1}^{3}\mu_1(s)f(s)(x), \sum_{s=1}^{3}\mu_2(s)f(s)(x) < \sum_{s=1}^{3}\mu_3(s)f(s)(x), \sum_{s=1}^{3}\mu_4(s)f(s)(x)$$

Hence, for all $f\in H$,

$$\{\mu_3,\mu_4\}\cap\left(\arg\min_{i\in\{1,3,4\}}\sum_{s=1}^{3}\mu_i(s)f(s)(x)\cup\arg\min_{i\in\{2,3,4\}}\sum_{s=1}^{3}\mu_i(s)f(s)(x)\right)\neq\varnothing$$

It follows that $W_{P_3}(f) = \min_{i\in\{3,4\}}\sum_{s=1}^{3}\mu_i(s)f(s)(x) \le \max\{W_{P_5}(f), W_{P_6}(f)\}$. Since $P_3 = P_5\cap P_6$, $W_{P_3}(f) \ge \max\{W_{P_5}(f), W_{P_6}(f)\}$, hence $W_{P_3}(f) = \max\{W_{P_5}(f), W_{P_6}(f)\}$ for all f, and by Lemma 11, P_3 is a coherent intersection of P_5 and P_6. ■

Online Appendix

In this Online Appendix, we provide additional details for the analysis done in the Appendix of the paper. Mostly, these are proofs that certain representations imply sets of axioms.

A. Omitted Proofs for Theorem 1 (Representation implies Axioms).

The proof that representation implies *Indirect Ambiguity Aversion* and *Continuity* is in the paper.

α : Let $f\in c(A\cup B)\cap A$, then $\exists P\in\mathscr{A}$: $W_P(f)\ge W_P(g)$ $\forall g\in A\subseteq A\cup B$, and $f\in c(A)$.

Aizerman's Property: Let $f\notin c(A\cup\{f\})$. Consider arbitrary $h\in c(A)$, then $\exists P\in\mathscr{A}$: $W_P(f)\ge W_P(g)$ $\forall g\in A$. If $W_P(f) > W_P(h)$, then $f\in c(A\cup\{f\})$, in contradiction. Hence, $W_P(h)\ge W_P(g)$ $\forall g\in A\cup\{f\}$. It follows $c(A)\subseteq c(A\cup\{f\})$.

C-β : Note that $W_P(p) = U(p)$ for all $p\in H_0$, $P\in\mathscr{A}\in\Pi$. Hence, $[W_P(p)\ge W_P(q)\forall P\in\mathscr{A}] \iff U(p)\ge U(q)$. Therefore, DM's choice on the domain of constant acts has utility representation U and, therefore, β holds on the domain of menus, consisting of constant acts.

C-Independence: Let $\lambda\in(0,1)$. Note that

$$W_P(\lambda f+(1-\lambda)p) = \min_{\mu\in P}\sum_{s\in S}\mu_s\cdot(\lambda U(f(s))+(1-\lambda)U(p)) = \tag{A.11}$$

$$= (1-\lambda)U(p)+\lambda\min_{\mu\in P}\sum_{s\in S}\mu_s\cdot U(f(s)) = \lambda W_P(f)+(1-\lambda)U(p)$$

Hence, $\forall P \in \mathscr{A}$ we have $[W_P(\lambda f + (1-\lambda)p) \geq W_P(\lambda g + (1-\lambda)p) \forall g \in A] \iff [\lambda W_P(f) + (1-\lambda)U(p) \geq \lambda W_P(g) + (1-\lambda)U(p) \forall g \in A] \iff [W_P(f) \geq W_P(g) \forall g \in A]$, and *C-Independence* follows.

No-C-Hedging: Let $f \in H$, $p \in H_0$, $f, p \in A$; consider arbitrary $h \in c(A)$. Then $\exists P \in \mathscr{A} : W_P(h) \geq W_P(g) \forall g \in A$, and in particular, $W_P(h) \geq W_P(f)$, $W_P(h) \geq U(p)$. It follows $W_P(h) \geq \lambda W_P(f) + (1-\lambda)U(p) = W_P(\lambda f + (1-\lambda)p)$. Therefore, $W_P(h) \geq W_P(g)$ $\forall g \in A \cup \{\lambda f + (1-\lambda)p\}$, and $h \in c(A \cup \{\lambda f + (1-\lambda)p\})$, proving *No-C-Hedging*.

Strict Monotonicity: $[g(s) \notin c(\{f(s), g(s)\}) \forall s \in S] \implies [U(g(s)) < U(f(s)) \forall s \in S] \implies [\sum_{s \in S} \mu_s \cdot U(g(s)) < \sum_{s \in S} \mu_s U(f(s)) \forall \mu \in \Pi] \implies [\min_{\mu \in P} \sum_{s \in S} \mu_s U(g(s)) < \min_{\mu \in P} \sum_{s \in S} \mu_s U(f(s)) \forall P \in \mathscr{A} \subseteq \Pi] \implies g \notin c(\{f, g\})$.

C-Non-Degeneracy: Since U is non-degenerate, $\exists p, q \in \triangle X$: $U(p) > U(q)$. Then $q \notin c(\{p, q\})$. ■

B. Omitted Proofs for Theorem 1 (Lemma 8: Duality).

Lemma 8. *Let $\mathscr{I}$ be the set of monotone positively homogeneous concave constant additive and normalized functions $I : \mathbb{R}^S \to \mathbb{R}$, and Π be the set of non-empty convex and closed sets of probabilities $P \subseteq \triangle S$. Then the mapping $\tau : \mathscr{I} \to \Pi$ given by $\tau(I) = \left\{\mu \in \triangle S \;\middle|\; \sum\limits_{s \in S} \mu(s) z_s \;\geq\; I(\mathbf{z}) \;\forall \mathbf{z} \in \mathbb{R}^S\right\}$ is a bijection, and $\tau^{-1}(P)(\mathbf{z}) \equiv \min\limits_{\mu \in P} \sum\limits_{s \in S} \mu(s) z_s$. Moreover, $W_{\tau(I_A)}(f) = I_A(f)$ $\forall f \in H$.*

Proof.

Claim B.1. *The mappings $\sigma(\cdot) : \Pi \to \mathscr{I}$ given by $\sigma(P)(\mathbf{z}) = \min\limits_{\mu \in P} \sum\limits_{s \in S} \mu(s) z_s$ and $\tau(\cdot)$ are well-defined.*

Proof of Claim B.1. One can see that if $P \in \Pi$, then indeed $\sigma(P) \in \mathscr{I}$. Now consider $I \in \mathscr{I}$. Note that $I(\mathbf{0}) = 0$ by positive homogeneity. Next, note that $I(\cdot)$ is continuous (moreover, Lipschitz continuous). Indeed, since $\mathbf{z} + \max\limits_{s \in S}\{z_s - z'_s\} \cdot \iota \geq \mathbf{z}'$, we get $|I(\mathbf{z}) - I(\mathbf{z}')| \leq |\mathbf{z} - \mathbf{z}'|$. Thus for arbitrary $\mathbf{z} \in \mathbb{R}^S$ the set $D(\mathbf{z}) = \left\{\tilde{\mathbf{z}} \in \mathbb{R}^S \;\middle|\; I(\tilde{\mathbf{z}}) \geq I(\mathbf{z})\right\}$ is closed. Next, $D(\mathbf{z})$ is convex, because $I(\cdot)$ is concave. Moreover, since $I(\cdot)$ is monotone, $\mathbf{z} - (1/n)\iota \notin D(\mathbf{z})$ for any $n = 1, 2, \ldots$, and $\mathbf{z} - (1/n)\iota \longrightarrow \mathbf{z}$, hence $\mathbf{z}$ is boundary point of $D(\mathbf{z})$, and by the Supporting Hyperplane Theorem there exist $\mu \in \mathbb{R}^S$ and $c \in \mathbb{R}$ such that $\sum\limits_{s \in S} \mu(s) z_s + c = I(\mathbf{z})$ and $\sum\limits_{s \in S} \mu(s) \tilde{z}_s + c \geq I(\tilde{\mathbf{z}})$ for all $\tilde{\mathbf{z}} \in \mathbb{R}^S$. Positive homogeneity of $I(\cdot)$ implies that $c \geq \alpha \left(\sum\limits_{s \in S} \mu(s) z_s - I(\mathbf{z})\right) = \alpha c$ for all $\alpha \geq 0$, hence $c = 0$. Next, assume $\mu(\overline{s}) < 0$ for some $\overline{s} \in S$. Consider $e_{\overline{s}} \in \mathbb{R}^S$ such that $(e_{\overline{s}})_s = \mathbb{1}\{s = \overline{s}\}$. Then, since $e_{\overline{s}} \geq \mathbf{0}$, we get $0 > \sum\limits_{s \in S} \mu(s)(e_{\overline{s}})_s \geq I(e_{\overline{s}}) \geq I(\mathbf{0}) = 0$, contradiction. Finally, $\sum\limits_{s \in S} \mu(s) \cdot 1 \geq I(\iota) = 1$ and $\sum\limits_{s \in S} \mu(s) \cdot (-1) \geq I(-\iota) = -1$ imply $\sum\limits_{s \in S} \mu(s) = 1$. Therefore, $\mu \in \triangle S$, and $\tau(I) \neq \varnothing$.

Next, function $\xi : \triangle S \times [-1,1]^S \to \mathbb{R}$ given by $\xi(\mu, \mathbf{z}) = \min_{s \in S} \mu(s) z_s - I(\mathbf{z})$ is continuous, therefore, by the Theorem of Maximum, function $\eta : \triangle S \to \mathbb{R}$ given by $\eta(\mu) = \min_{\mathbf{z} \in [-1,1]^S} \xi(\mu, \mathbf{z})$ is continuous. Take a sequence $\mu^n \in \tau(I)$ that converges to some $\mu \in \triangle S$. Then $\eta(\mu^n) \geq 0$ for all n, and we conclude that $\eta(\mu) \geq 0$. By positive homogeneity of $I(\cdot)$, we get $\sum_{s \in S} \mu(s) z_s - I(\mathbf{z}) \geq 0 \ \forall \mathbf{z} \in [-1,1]^S \iff \sum_{s \in S} \mu(s) z_s - I(\mathbf{z}) \geq 0 \ \forall \mathbf{z} \in \mathbb{R}^S$, thus $\mu \in \tau(I)$, hence $\tau(I)$ is closed. Finally, $\tau(I)$ is obviously convex, thus $\tau(I) \in \Pi$, proving the claim. $\square$

Claim B.2. *$\sigma(\tau(I)) = I$ for all $I \in \mathscr{I}$.*

Proof of Claim B.2. One can see that $\sigma(\tau(I))(\mathbf{z}) \geq I(\mathbf{z})$ for all $\mathbf{z} \in \mathbb{R}^S$, since $\sum_{s \in S} \mu(s) z_s \geq I(\mathbf{z})$ for all $\mu \in \tau(I)$ for all $\mathbf{z} \in \mathbb{R}^S$. From the other hand, by the analysis in the proof of Claim B.1, for all $\mathbf{z} \in \mathbb{R}^S$ there is $\mu \in \tau(I)$ such that $\sum_{s \in S} \mu(s) z_s = I(\mathbf{z})$. Thus, $\sigma(\tau(I))(\mathbf{z}) \leq I(\mathbf{z})$ for all $\mathbf{z} \in \mathbb{R}^S$. We conclude that $\sigma(\tau(I)) = I$. $\square$

Claim B.3. *$\tau(\sigma(P)) = P$ for all $P \in \Pi$.*

Proof of Claim B.3. Take any $\mu \in P$, then $\sum_{s \in S} \mu(s) z_s \geq \min_{\mu' \in P} \sum_{s \in S} \mu'(s) z_s = I(\mathbf{z})$ for any $\mathbf{z} \in \mathbb{R}^S$, thus $P \subseteq \tau(\sigma(P))$. Towards a contradiction, assume that there exists $\mu \in \tau(\sigma(P)) \setminus P \neq \varnothing$. Since $\{\mu\}$ and P are disjoint compact subsets of $\mathbb{R}^S$, by the Separating Hyperplane Theorem, there exists $\mathbf{z} \in \mathbb{R}^S$ such that $\sum_{s \in S} \mu(s) z_s < c < \min_{\mu' \in P} \sum_{s \in S} \mu'(s) z_s = I(\mathbf{z})$, contradiction. Thus, $\tau(\sigma(P)) \subseteq P$, proving the claim. $\square$

Claims B.1, B.2, B.3 prove the Lemma. $\blacksquare$

C. Omitted Proofs for Proposition 1.

In the paper, we present a series of examples when one axiom is violated, while all other axioms hold. Here, we give proofs that all other axioms hold for each of the examples.

Independence of Framed Uncertainty. Recall the choice correspondence used in the proof:

$$c(A) = \left\{ f \in A \,\middle|\, \sum_{s \in S} f(s)(x) \geq \sum_{s \in S} g(s)(x) \ \forall g \in A \text{ or } \sum_{s \in S} f(s)(y) \geq \sum_{s \in S} g(s)(y) \ \forall g \in A \right\}$$

Denote by $U, V : H \to \mathbb{R}$ given by $U(f) = \sum_{s \in S} f(s)(x)$ and $V(f) = \sum_{s \in S} f(s)(y)$. Then

$$c(A) = c_U(A) \cup c_V(A)$$

where $c_U(A) = \left\{ f \in A \middle| U(f) \geq U(g) \ \forall g \in A \right\}$, $c_V(A) = \left\{ f \in A \middle| V(f) \geq V(g) \ \forall g \in A \right\}$. Thus, U and V are utility functions of subjective expected utility maximizers with uniform prior and vNM expect utility indexes given by $u(p) = p(x)$, $v(p) = p(y)$ correspondingly; $c_U(\cdot)$ and $c_V(\cdot)$ are the induced choice correspondences. It follows $c_U(\lambda A + (1-\lambda)g) = \lambda c_U(A) + (1-\lambda)g$, and $c_V(\lambda A + (1-\lambda)g) = \lambda c_V(A) + (1-\lambda)g$, hence $c(\lambda A + (1-\lambda)g) = \lambda c(A) + (1-\lambda)g$. In particular, *C-Independence* holds.

Next, let $h \in c(A)$, $f, p \in A$, then either $U(h) \geq U(f), U(p)$ and hence, $U(h) \geq \lambda U(f) + (1-\lambda)U(p) = U(\lambda f + (1-\lambda)p)$, $h \in c_U(A \cup \{\lambda f + (1-\lambda)p\})$, or $V(h) \geq V(f), V(p)$ and hence, $V(h) \geq \lambda V(f) + (1-\lambda)V(p) = V(\lambda f + (1-\lambda)p)$, $h \in c_V(A \cup \{\lambda f + (1-\lambda)p\})$, or both. Therefore, $h \in c(A \cup \{\lambda f + (1-\lambda)p\})$, proving *No-C-Hedging*.

Assume $g(s) \notin c(\{f(s), g(s)\})$ $\forall s \in S$. Then $g(s)(x) < f(s)(x)$ and $g(s)(y) < f(s)(y)$ $\forall s \in S$, hence $U(g) < U(f)$ and $V(g) < V(f)$, implying $g \notin c(\{f, g\})$. *Strict Monotonicity* holds.

Let $h \notin A \cup \{f\}$, $h \notin A \cup \{g\}$. If $U(h) \geq U(h')$ $\forall h' \in A$, then $U(h) < U(f), U(g)$, and $U(h) < \lambda U(f) + (1-\lambda)U(g) = U(\lambda f + (1-\lambda)g)$; it follows that in any case, $h \notin c_U(A \cup \{\lambda f + (1-\lambda)g\})$. Similarly, if $V(h) \geq V(h')$ $\forall h' \in A$, then $V(h) < V(f), V(g)$, and $V(h) < \lambda V(f) + (1-\lambda)V(g) = V(\lambda f + (1-\lambda)g)$, and in any case, $h \notin c_V(A \cup \{\lambda f + (1-\lambda)g\})$. Therefore, $h \notin c(A \cup \{\lambda f + (1-\lambda)g\})$. *Indirect Ambiguity Aversion* holds.

Since U, V are continuous, $\{(f, A) | f \in c_U(A)\}$ and $\{(f, A) | f \in c_V(A)\}$ are closed. Hence, $\{(f, A) | f \in c(A)\} = \{(f, A) | f \in c_U(A)\} \cup \{(f, A) | f \in c_V(A)\}$ is closed. *Continuity* holds.

Finally, $z \notin c(\{0.5x + 0.5y, z\})$, thus *C-Non-Degeneracy* holds. □

Independence of *C-Independence*. Recall the choice correspondence used in the proof:

$$c(A) \;=\; \arg\max_{f \in A} W(f), \qquad W(f) \;\equiv\; \frac{f(1)(x) + f(2)(x)}{1 + f(2)(x)} = 1 - \frac{1 - f(1)(x)}{1 + f(2)(x)}$$

Since $c(\cdot)$ has continuous utility representation, *WARP*, and, hence, *Framed Ambiguity* holds, and also *Continuity* holds.

Next, note that for all numbers a, b, c, d such that $b, d > 0$ and $a, c \geq 0$, and $\lambda \in [0, 1]$,

$$\frac{a}{b} \leq \frac{c}{d} \quad \Longrightarrow \quad \frac{a}{b} \;\leq\; \frac{\lambda a + (1-\lambda)c}{\lambda b + (1-\lambda)d} \;\leq\; \frac{c}{d} \tag{C.12}$$

With $a = f(1)(x) + f(2)(x)$, $b = 1 + f(2)(x)$, $c = g(1)(x) + g(2)(x)$, $d = 1 + g(2)(x)$ for the case when $W(f) \geq W(g)$, and similar substitution for the other case, we get

$$\min\{W(f), W(g)\} \leq W(\lambda f + (1-\lambda)g) \leq \max\{W(f), W(g)\}$$

If $h \in c(A)$, $f, p \in A$, then $W(\lambda f + (1-\lambda)g) \leq \max\{W(f), W(g)\} \leq W(h)$, hence $h \in c(A \cup \{\lambda f + (1-$

$\lambda)p\}$), proving *No-C-Hedging*.

Note that $W(p) \geq W(q)$ if and only if $p(x) \geq q(x)$. Hence, if $g(s) \notin c(\{f(s), g(s)\})$ $\forall s \in S$, then $f(1)(x) > g(1)(x)$ and $f(2)(x) > g(2)(x)$. Since $W(\cdot)$ is strictly monotone with respect to $f(1)(x)$ and $f(2)(x)$, $W(f) > W(g)$, hence $g \notin c(\{f, g\})$. *Strict Monotonicity* holds.

Let $h \in A$, $h \notin A \cup \{f\}$, $h \notin A \cup \{g\}$. Then either $W(h') > W(h)$ for $h' \in A$, in which case $h \in c(A \cup \{\lambda f + (1-\lambda)p\})$, or $W(f) > W(h)$ and $W(g) > W(h)$. In this case, $W(h) < \min\{W(f), W(g)\} \leq W(\lambda f + (1-\lambda)g)$, and again, $h \in c(A \cup \{\lambda f + (1-\lambda)p\})$. Thus, *Indirect Ambiguity Aversion* holds.

Finally, since $W(y) = 0 < 1 = W(x)$, *C-Non-Degeneracy* holds. □

Independence of *No-C-Hedging*. Recall choice correspondence $c(\cdot)$ used in the proof. Let $x \in X$, $0 < \underline{\mu} \leq \overline{\mu} \leq 1$. Then

$$c(B) = \bigcup_{k \in [0,\infty)} \arg\max_{f \in B} W_k(f),$$

where

$$W_k(f) = \min\left\{(1-\overline{\mu})f(1)(x) + \overline{\mu}f(2)(x), (1-\underline{\mu})f(1)(x) + \underline{\mu}f(2)(x), f(1)(x) + \underline{\mu}k\right\}$$

Let $f \in c(A \cup B) \cap A$, then $\exists k \geq 0$: $W_k(f) \geq W_k(g)$ $\forall g \in A \subseteq A \cup B$, hence $f \in c(A)$. Thus, α holds.

Let $f \notin c(A \cup \{f\})$. Consider arbitrary $h \in c(f)$, then $\exists k \geq 0$: $W_k(f) \geq W_k(g)$ $\forall g \in A$. If $W_k(f) > W_k(h)$, then $f \in c(A \cup \{f\})$, in contradiction. Hence, $W_k(h) \geq W_k(g)$ $\forall g \in A \cup \{f\}$. It follows $c(A) \subseteq c(A \cup \{f\})$. Hence, *Aizerman's Property* holds.

Note that $W(p) = p(x)$ for all $p \in H_0$. Therefore, *C-β* holds. Also, $y \notin c(\{x, y\})$ for $y \in X \setminus \{x\}$, proving *C-Non-Degeneracy*.

Note that $W_k(\lambda f + (1-\lambda)p) = \lambda W_{k/\lambda}(f) + (1-\lambda)p(x)$ for all $p \in H_0$, $f \in H$, $k \geq 0$, $\lambda > 0$. Thus, $W_k(\lambda f + (1-\lambda)p) \geq W_k(\lambda g + (1-\lambda)p) \iff \lambda W_{k/\lambda}(f) + (1-\lambda)p(x) \geq \lambda W_{k/\lambda}(g) + (1-\lambda)p(x) \iff W_{k/\lambda}(f) \geq W_{k/\lambda}(g)$. To see that *C-Independence* holds, note that, using $k' = k/\lambda$, we get

$$\exists k \geq 0 : W_k(\lambda f + (1-\lambda)p) \geq W_k(\lambda g + (1-\lambda)p) \forall g \in A \iff \exists k' \geq 0 : W_{k'}(f) \geq W_{k'}(g) \forall g \in A$$

Assume $g(s) \notin c(\{f(s), g(s)\})$ $\forall s \in S$. Then, $g(1)(x) < f(1)(x)$ and $g(2)(x) < f(2)(x)$. It follows $W_k(g) < W_k(f)$ for all $k \geq 0$, hence, $g \notin c(\{f, g\})$. *Strict Monotonicity* holds.

Next, let $h \in A$, $h \notin A \cup \{f\}$, $h \notin A \cup \{g\}$. Note that

$$\min\{W_k(f), W_k(g)\} = \min\Big\{\min\left\{(1-\overline{\mu})f(1)(x) + \overline{\mu}f(2)(x), (1-\underline{\mu})f(1)(x) + \underline{\mu}f(2)(x), f(1)(x) + \underline{\mu}k\right\},$$
$$\min\left\{(1-\overline{\mu})g(1)(x) + \overline{\mu}g(2)(x), (1-\underline{\mu})g(1)(x) + \underline{\mu}g(2)(x), g(1)(x) + \underline{\mu}k\right\}\Big\} =$$

$$= \min\Big\{\min\left\{(1-\overline{\mu})f(1)(x)+\overline{\mu}f(2)(x),(1-\overline{\mu})g(1)(x)+\overline{\mu}g(2)(x)\right\},$$

$$\min\left\{(1-\underline{\mu})f(1)(x)+\underline{\mu}f(2)(x),(1-\underline{\mu})g(1)(x)+\underline{\mu}g(2)(x)\right\},\min\left\{f(1)(x)+\underline{\mu}k,g(1)(x)+\underline{\mu}k\right\}\Big\} \leq$$

$$\leq \min\Big\{(1-\overline{\mu})(\lambda f(1)(x)+(1-\lambda)g(1)(x))+\overline{\mu}(\lambda f(2)(x)+(1-\lambda)g(2)(x)),$$

$$(1-\underline{\mu})(\lambda f(1)(x)+(1-\lambda)g(1)(x)+\underline{\mu}(\lambda f(2)(x)+(1-\lambda)g(2)(x)),\lambda f(1)(x)+(1-\lambda)g(1)(x)+\underline{\mu}k\Big\} =$$

$$= W_k(\lambda f+(1-\lambda)g)$$

let $\mathbb{K} = \{k \geq 0 | W_k(h) \geq W_k(h') \forall h' \in A\}$, then $W_k(h) < \min\{W_k(f), W_k(g)\} \leq W_k(\lambda f+(1-\lambda)g)$ for all $k \in \mathbb{K}$. Thus, for all $k \geq 0$, $\exists h' \in A \cup \{\lambda f+(1-\lambda)p\} : W_k(h') > W_k(h)$. Hence, $h \notin c(A \cup \{\lambda f + (1-\lambda)p\})$, proving *Indirect Ambiguity Aversion.*

Let $(A^l, f^l) \longrightarrow (A, f) : f^l \in c(A^l)\ \forall l = 1,2,...$ Then $f^l \in A^l$, and $\rho(\{f\}, A) \leq d(f, f^l) + \rho(\{f^l\}, A) \leq d(f, f^l) + \rho(A^l, A) \longrightarrow 0$, hence $f \in cl(A) = A$. Note that $W_k(f) = W_1(f)$ for all $k > 1$. Therefore,

$$c(B) = \bigcup_{k \in [0,\infty)} \arg\max_{f \in B} W_k(f) = \bigcup_{k \in [0,1]} \arg\max_{f \in B} W_k(f)$$

Since $f^l \in c(A^l)$ for $l = 1,2,...$, $\exists k^l \in [0,1]$: $W_{k^l}(f^l) \geq W_{k^l}(h)\ \forall h \in A^l$. Since $[0,1]$ is compact, $\exists k^{l_m} \longrightarrow k \in [0,1]$. Consider arbitrary $g \in A$; since $A^{l_m} \longrightarrow A$, $\exists g^{l_m} \in A^{l_m}$: $g^{l_m} \longrightarrow g$. Then $W_{P^{l_m}}(f^{l_m}) \geq W_{P^{l_m}}(g^{l_m})\ \forall m$. Note that $(j, h, h') \to W_j(h) - W_j(h')$ is continuous, hence $W_k(f) \geq W_k(g)$. Therefore, $f \in c(A)$. *Continuity* holds. □

Independence of *Strict Monotonocity.* Recall choice correspondence $c(A) = \arg\max_{f \in A} W(f)$, $W(f) = 2f(1)(x) - f(2)(x)$ used in the proof.

Since $c(\cdot)$ has continuous utility representation, *WARP*, and, hence, *Framed Ambiguity* holds, and also *Continuity* holds. Next, since $W(\lambda f+(1-\lambda)g) = \lambda W(f)+(1-\lambda)W(g)$, *C-Independence*, *No-C-Hedging*, and *Indirect Ambiguity Aversion* hold. Finally, $W(x) = 1$, $W(y) = 0$ for $y \in X \setminus \{x\}$, hence $y \notin c(\{x, y\})$, and *C-Non-Degeneracy* holds. □

Independence of Indirect Ambiguity Aversion. Recall choice correspondence $c(A) = \arg\max_{f \in A} W(f)$ with $W(f) = \max\{f(1)(x), f(2)(x)\}$ used in the proof.

Since $c(\cdot)$ has continuous utility representation, *WARP*, and, hence, *Framed Ambiguity* holds, and also *Continuity* holds. Since $W(\lambda f + (1-\lambda)p) = \lambda W(f) + (1-\lambda)p(x)$ for all $f \in H$ $p \in H_0$, *C-Independence* and *No-C-Hedging* hold. Next, if $g(s) \notin c(\{f(s), g(s)\})\ \forall s \in S$, then $f(1)(x) > g(1)(x)$ and $f(2)(x) > g(2)(x)$, hence $W(f) > W(g)$. Therefore, *Strict Monotonicity* holds. Finally, $W(x) =$

$1 > 0 = W(y)$ for $y \in X \setminus \{x\}$, hence *C-Non-Degeneracy* holds. $\square$

Independence of *Continuity*. Recall choice correspondence $c(A) = \{f \in A | f \succeq g \ \forall g \in A\}$ with $f \succeq g$ if either $f(1)(x) > g(1)(x)$, or $[f(1)(x) = g(1)(x)$ and $f(2)(x) \geq g(2)(x)]$ used in the proof.

Since $\succeq$ is complete and transitive, $c(\cdot)$ satisfies *WARP* and, hence *Framed Uncertainty*. Since for any $\lambda \in (0,1)$, any $f, g, h \in H$ we have $f \succeq g \iff \lambda f + (1-\lambda)h \succeq \lambda g + (1-\lambda)h$, C-Independece, *No-C-Hedging*, and *Indirect Ambiguity Aversion* hold for $c(\cdot)$. Next, $g(s) \notin c(\{f(s), g(s)\})$ implies $g(1) \prec f(1)$, hence $g(1)(x) < f(1)(x)$, $g \prec f$, and $g \notin c(\{f, g\})$, proving *Strict Monotonicity*. Finally, $x \succ y$ for $y \in X \setminus \{x\}$, which yields *C-Non-Degeneracy* for $c(\cdot)$.

Independence of *C-Non-Degeneracy*. Choice correspondence $c(A) = A$ considered in the proof satisfies *Framed Uncertainty*, *C-Independence*, *No-C-Hedging* because the conclusions of these axioms are always satisfied for $c(\cdot)$ independent of the premise. Similarly, $c(\cdot)$ satisfies *Strict Monotonicity* and *Indirect Ambiguity Aversion* because the premises of these axioms are never satisfied. Finally, $\{(A, f) \in \mathscr{K} \times H | f \in c(A)\} = \mathscr{K} \times H$ is closed, hence $c(\cdot)$ is continuous. $\square$

D. Omitted Proofs for Lemma 27.

Recall choice correspondence $c(A) = \left\{f \in A \middle| f(1)(x) \geq f(2)(x) \text{ or } f(2)(x) \geq g(2)(x) \ \forall g \in A\right\}$ considered in Lemma 27. In the paper, we show that $c(\cdot)$ satisfies $c(A) \neq A$ for some $A \in \mathscr{K}$, $c(\{f, g\}) \neq \{f, g\}$ for some $f, g \in H$, $c(A) = A$ for all $A \in \mathscr{K}_0$ (and in particular, $c(\cdot)$ violates *C-Non-Degeneracy*). Here, we show that $c(\cdot)$ satisfies Axioms 1–6.

Let $f \in c(A \cup B) \cap A$. If $f(1) \geq f(2)$, then $f \in c(A)$. If $f(1) < f(2)$, then $f(2) \geq g(2)$ for all $g \in A \subseteq A \cup B$, thus $f \in c(A)$. Hence, α holds.

Let $h \in c(A)$. If $h(1) \geq h(2)$, then $h \in c(A \cup \{f\})$. Otherwise $h(2) \geq g(2)$ for all $g \in A$. Then if $h(2) \geq f(2)$, then $h \in c(A \cup \{f\})$. Otherwise $f(2) > h(2) \geq g(2)$ for all $g \in A$, hence $f \in c(A \cup \{f\})$, in contradiction. Therefore, *Aizerman's Property* holds.

Since $c(A) = A$ for all $A \in \mathscr{K}_0$, *C-β* holds.

For $p \in H_0$ denote by $f_\lambda = \lambda f + (1-\lambda)p$. Since $f_\lambda(1) \geq f_\lambda(2) \iff f(1) \geq f(2)$ and $[f_\lambda(2) \geq g_\lambda(2) \ \forall g_\lambda \in \lambda A + (1-\lambda)p] \iff [f_\lambda(2) \geq g_\lambda(2) \ \forall g \in A]$, *C-Independence* holds.

Let $h \in c(A)$ with $f, p \in A$. If $h(1)(x) \geq h(2)(x)$, then $h \in c(A \cup \{\lambda f + (1-\lambda)p\})$. Otherwise, $h(2)(x) \geq g(2)(x)$ for all $g \in A$, and in particular, $h(2)(x) \geq f(2)(x), p(x)$. Therefore, $h(2)(x) \geq \lambda f(2)(x) + (1-\lambda)p(2)(x)$, and hence, $h(2)(x) \geq g(2)(x)$ for all $g \in A \cup \{\lambda f + (1-\lambda)p\}$, and $h \in$

$c(A \cup \{\lambda f + (1-\lambda)p\})$, proving *No-C-Hedging*.

Since $c(\{f(s), g(s)\}) = \{f(s), g(s)\}$ for any $f, g \in H$, $s \in S$, *Strict Monotonicity* holds[28].

Consider $h \in A$, $h \notin c(A \cup \{f\})$, $h \notin c(A \cup \{g\}$. If $h \notin c(A)$, by the proven condition α, we have $h \notin c(A \cup \{\lambda f + (1-\lambda)g\})$. Let $h \in c(A)$. If $h(1)(x) \geq h(2)(x)$, then $h \in c(A \cup \{f\})$, in contradiction. Hence, $h(1)(x) < h(2)(x)$, and $h(2)(x) < f(2)(x), g(2)(x)$. Thus, $h(2)(x) < \lambda f(2)(x) + (1-\lambda)g(2)(x)$. Since also $h(1)(x) < h(2)(x)$, then $h \notin c(A \cup \{\lambda f + (1-\lambda)g\})$, proving *Indirect Ambiguity Aversion.*

Let $(A^n, f^n) \to (A, f)$ be such that $f^n \in c(A^n)$ for all $n = 1, 2, \ldots$. If $f(1)(x) \geq f(2)(x)$, then $f \in c(A)$. Otherwise, $f(1)(x) < f(2)(x)$ and $f^n(1)(x) < f^n(2)(x)$ for large enough n; WLOG this is true all n. Therefore, $f^n(2)(x) \geq g^n(2)(x)$ for all $g^n \in A^n$. Consider arbitrary $g \in A$, then $\exists g^n \to g$: $g^n \in A^n$ for all n. Note that the function $u : \mathcal{K} \times H \to R$ given by $u(A, h) = h(2)(x) - \max_{g \in A} g(2)(x)$ is continuous; therefore, since $f^n(2)(x) - \max_{g \in A^n} g(2)(x) \geq 0$ for all n, then $f(2)(x) - \max_{g \in A} g(2)(x) \geq 0$. Hence, $f \in c(A)$, proving *Continuity.*

Notice that function $v : \mathcal{K} \to R$ given by $v(A) = \max_{g \in A} g(2)$ is continuous, thus function $u : \mathcal{K} \times H \to R$ given by $u(h, A) = h(2) - \max_{g \in A} g(2)$ is continuous. Hence, $f^n(2) - \max_{g \in A^n} g(2) = 0$ for all large enough n implies $f(2) - \max_{g \in A} g(2) = 0$ in the limit. It follows that $f(2) \geq g(2)$ for all $g \in A$, hence $f \in c(A)$, proving *Continuity.* ■

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