Online Appendix for "Framing of Ambiguity"

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In this Online Appendix, we provide additional details for the analysis done in the Appendix of the paper. Mostly, these are proofs that certain representations imply sets of axioms.

A. Omitted Proofs for Theorem 1 (Representation implies Axioms).

The proof that representation implies *Indirect Ambiguity Aversion* and *Continuity* is in the paper.

α : Let $f \in c(A \cup B) \cap A$, then $\exists P \in \mathscr{A}$: $W_P(f) \geq W_P(g)$ $\forall g \in A \subseteq A \cup B$, and $f \in c(A)$.

Aizerman's Property: Let $f \notin c(A \cup \{f\})$. Consider arbitrary $h \in c(A)$, then $\exists P \in \mathscr{A}$: $W_P(f) \geq W_P(g)$ $\forall g \in A$. If $W_P(f) > W_P(h)$, then $f \in c(A \cup \{f\})$, in contradiction. Hence, $W_P(h) \geq W_P(g)$ $\forall g \in A \cup \{f\}$. It follows $c(A) \subseteq c(A \cup \{f\})$.

C-β : Note that $W_P(p) = U(p)$ for all $p \in H_0$, $P \in \mathscr{A} \in \Pi$. Hence, $[W_P(p) \geq W_P(q) \forall P \in \mathscr{A}] \iff U(p) \geq U(q)$. Therefore, DM's choice on the domain of constant acts has utility representation U and, therefore, β holds on the domain of menus, consisting of constant acts.

C-Independence: Let $\lambda \in (0,1)$. Note that

$$W_P(\lambda f + (1-\lambda)p) = \min_{\mu \in P} \sum_{s \in S} \mu_s \cdot (\lambda U(f(s)) + (1-\lambda)U(p)) = \tag{A.1}$$
$$= (1-\lambda)U(p) + \lambda \min_{\mu \in P} \sum_{s \in S} \mu_s \cdot U(f(s)) = \lambda W_P(f) + (1-\lambda)U(p)$$

Hence, $\forall P \in \mathscr{A}$ we have $[W_P(\lambda f + (1-\lambda)p) \geq W_P(\lambda g + (1-\lambda)p) \forall g \in A] \iff [\lambda W_P(f) + (1-\lambda)U(p) \geq \lambda W_P(g) + (1-\lambda)U(p) \forall g \in A] \iff [W_P(f) \geq W_P(g) \forall g \in A]$, and *C-Independence* follows.

No-C-Hedging: Let $f \in H$, $p \in H_0$, $f, p \in A$; consider arbitrary $h \in c(A)$. Then $\exists P \in \mathscr{A} : W_P(h) \geq W_P(g) \forall g \in A$, and in particular, $W_P(h) \geq W_P(f)$, $W_P(h) \geq U(p)$. It follows $W_P(h) \geq \lambda W_P(f) + (1-\lambda)U(p) = W_P(\lambda f + (1-\lambda)p)$. Therefore, $W_P(h) \geq W_P(g)$ $\forall g \in A \cup \{\lambda f + (1-\lambda)p\}$, and $h \in c(A \cup \{\lambda f + (1-\lambda)p\})$, proving *No-C-Hedging*.

Strict Monotonicity: $[g(s) \notin c(\{f(s), g(s)\}) \forall s \in S] \implies [U(g(s)) < U(f(s)) \forall s \in S] \implies [\sum_{s \in S} \mu_s \cdot U(g(s)) < \sum_{s \in S} \mu_s U(f(s)) \forall \mu \in \Pi] \implies [\min_{\mu \in P} \sum_{s \in S} \mu_s U(g(s)) < \min_{\mu \in P} \sum_{s \in S} \mu_s U(f(s)) \forall P \in \mathscr{A} \subseteq \Pi] \implies g \notin c(\{f, g\})$.

C-Non-Degeneracy: Since U is non-degenerate, $\exists p, q \in \triangle X$: $U(p) > U(q)$. Then $q \notin c(\{p, q\})$. ■

B. Omitted Proofs for Theorem 1 (Lemma 8: Duality).

Lemma 8. *Let $\mathscr{I}$ be the set of monotone positively homogeneous concave constant additive and normalized functions $I: \mathbb{R}^S \to \mathbb{R}$, and Π be the set of non-empty convex and closed sets of probabilities $P \subseteq \triangle S$. Then the mapping $\tau: \mathscr{I} \to \Pi$ given by $\tau(I) = \left\{\mu \in \triangle S \;\middle|\; \sum_{s\in S} \mu(s) z_s \geq I(\mathbf{z}) \; \forall \mathbf{z} \in \mathbb{R}^S\right\}$ is a bijection, and $\tau^{-1}(P)(\mathbf{z}) \equiv \min_{\mu\in P} \sum_{s\in S} \mu(s) z_s$. Moreover, $W_{\tau(I_A)}(f) = I_A(f) \; \forall f \in H$.*

Proof.

Claim B.1. *The mappings $\sigma(\cdot): \Pi \to \mathscr{I}$ given by $\sigma(P)(\mathbf{z}) = \min_{\mu\in P} \sum_{s\in S} \mu(s) z_s$ and $\tau(\cdot)$ are well-defined.*

Proof of Claim B.1. One can see that if $P \in \Pi$, then indeed $\sigma(P) \in \mathscr{I}$. Now consider $I \in \mathscr{I}$. Note that $I(\mathbf{0}) = 0$ by positive homogeneity. Next, note that $I(\cdot)$ is continuous (moreover, Lipschitz continuous). Indeed, since $\mathbf{z} + \max_{s\in S}\{z_s - z'_s\} \cdot \iota \geq \mathbf{z}'$, we get $|I(\mathbf{z}) - I(\mathbf{z}')| \leq |\mathbf{z} - \mathbf{z}'|$. Thus for arbitrary $\mathbf{z} \in \mathbb{R}^S$ the set $D(\mathbf{z}) = \{\tilde{\mathbf{z}} \in \mathbb{R}^S \mid I(\tilde{\mathbf{z}}) \geq I(\mathbf{z})\}$ is closed. Next, $D(\mathbf{z})$ is convex, because $I(\cdot)$ is concave. Moreover, since $I(\cdot)$ is monotone, $\mathbf{z} - (1/n)\iota \notin D(\mathbf{z})$ for any $n = 1, 2, \ldots$, and $\mathbf{z} - (1/n)\iota \longrightarrow \mathbf{z}$, hence $\mathbf{z}$ is boundary point of $D(\mathbf{z})$, and by the Supporting Hyperplane Theorem there exist $\mu \in \mathbb{R}^S$ and $c \in \mathbb{R}$ such that $\sum_{s\in S} \mu(s) z_s + c = I(\mathbf{z})$ and $\sum_{s\in S} \mu(s) \tilde{z}_s + c \geq I(\tilde{\mathbf{z}})$ for all $\tilde{\mathbf{z}} \in \mathbb{R}^S$. Positive homogeneity of $I(\cdot)$ implies that $c \geq \alpha \left(\sum_{s\in S} \mu(s) z_s - I(\mathbf{z})\right) = \alpha c$ for all $\alpha \geq 0$, hence $c = 0$. Next, assume $\mu(\bar{s}) < 0$ for some $\bar{s} \in S$. Consider $e_{\bar{s}} \in \mathbb{R}^S$ such that $(e_{\bar{s}})_s = \mathbb{1}\{s = \bar{s}\}$. Then, since $e_{\bar{s}} \geq \mathbf{0}$, we get $0 > \sum_{s\in S} \mu(s)(e_{\bar{s}})_s \geq I(e_{\bar{s}}) \geq I(\mathbf{0}) = 0$, contradiction. Finally, $\sum_{s\in S} \mu(s) \cdot 1 \geq I(\iota) = 1$ and $\sum_{s\in S} \mu(s) \cdot (-1) \geq I(-\iota) = -1$ imply $\sum_{s\in S} \mu(s) = 1$. Therefore, $\mu \in \triangle S$, and $\tau(I) \neq \varnothing$.

Next, function $\xi: \triangle S \times [-1,1]^S \to \mathbb{R}$ given by $\xi(\mu, \mathbf{z}) = \min_{s\in S} \mu(s) z_s - I(\mathbf{z})$ is continuous, therefore, by the Theorem of Maximum, function $\eta: \triangle S \to \mathbb{R}$ given by $\eta(\mu) = \min_{\mathbf{z}\in[-1,1]^S} \xi(\mu, \mathbf{z})$ is continuous. Take a sequence $\mu^n \in \tau(I)$ that converges to some $\mu \in \triangle S$. Then $\eta(\mu^n) \geq 0$ for all n, and we conclude that $\eta(\mu) \geq 0$. By positive homogeneity of $I(\cdot)$, we get $\sum_{s\in S} \mu(s) z_s - I(\mathbf{z}) \geq 0 \; \forall \mathbf{z} \in [-1,1]^S \iff \sum_{s\in S} \mu(s) z_s - I(\mathbf{z}) \geq 0 \; \forall \mathbf{z} \in \mathbb{R}^S$, thus $\mu \in \tau(I)$, hence $\tau(I)$ is closed. Finally, $\tau(I)$ is obviously convex, thus $\tau(I) \in \Pi$, proving the claim. □

Claim B.2. *$\sigma(\tau(I)) = I$ for all $I \in \mathscr{I}$.*

Proof of Claim B.2. One can see that $\sigma(\tau(I))(\mathbf{z}) \geq I(\mathbf{z})$ for all $\mathbf{z} \in \mathbb{R}^S$, since $\sum_{s\in S} \mu(s) z_s \geq I(\mathbf{z})$ for all $\mu \in \tau(I)$ for all $\mathbf{z} \in \mathbb{R}^S$. From the other hand, by the analysis in the proof of Claim B.1, for all $\mathbf{z} \in \mathbb{R}^S$ there is $\mu \in \tau(I)$ such that $\sum_{s\in S} \mu(s) z_s = I(\mathbf{z})$. Thus, $\sigma(\tau(I))(\mathbf{z}) \leq I(\mathbf{z})$ for all $\mathbf{z} \in \mathbb{R}^S$. We conclude that $\sigma(\tau(I)) = I$. $\square$

Claim B.3. $\tau(\sigma(P)) = P$ *for all* $P \in \Pi$.

Proof of Claim B.3. Take any $\mu \in P$, then $\sum_{s\in S} \mu(s)z_s \geq \min_{\mu'\in P} \sum_{s\in S} \mu'(s)z_s = I(\mathbf{z})$ for any $\mathbf{z} \in \mathbb{R}^S$, thus $P \subseteq \tau(\sigma(P))$. Towards a contradiction, assume that there exists $\mu \in \tau(\sigma(P))\backslash P \neq \varnothing$. Since $\{\mu\}$ and P are disjoint compact subsets of $\mathbb{R}^S$, by the Separating Hyperplane Theorem, there exists $\mathbf{z} \in \mathbb{R}^S$ such that $\sum_{s\in S} \mu(s)z_s < c < \min_{\mu'\in P} \sum_{s\in S} \mu'(s)z_s = I(\mathbf{z})$, contradiction. Thus, $\tau(\sigma(P)) \subseteq P$, proving the claim. $\square$

Claims B.1, B.2, B.3 prove the Lemma. $\blacksquare$

C. Omitted Proofs for Proposition 1.

In the paper, we present a series of examples when one axiom is violated, while all other axioms hold. Here, we give proofs that all other axioms hold for each of the examples.

Independence of Framed Uncertainty. Recall the choice correspondence used in the proof:

$$c(A) \;=\; \left\{ f \in A \;\middle|\; \sum_{s\in S} f(s)(x) \geq \sum_{s\in S} g(s)(x) \;\forall g \in A \text{ or } \sum_{s\in S} f(s)(y) \geq \sum_{s\in S} g(s)(y) \;\forall g \in A \right\}$$

Denote by $U, V : H \to \mathbb{R}$ given by $U(f) = \sum_{s\in S} f(s)(x)$ and $V(f) = \sum_{s\in S} f(s)(y)$. Then

$$c(A) \;=\; c_U(A) \cup c_V(A)$$

where $c_U(A) = \left\{f \in A \middle| U(f) \geq U(g) \;\forall g \in A\right\}$, $c_V(A) = \left\{f \in A \middle| V(f) \geq V(g) \;\forall g \in A\right\}$. Thus, U and V are utility functions of subjective expected utility maximizers with uniform prior and vNM expect utility indexes given by $u(p) = p(x)$, $v(p) = p(y)$ correspondingly; $c_U(\cdot)$ and $c_V(\cdot)$ are the induced choice correspondences. It follows $c_U(\lambda A + (1-\lambda)g) = \lambda c_U(A) + (1-\lambda)g$, and $c_V(\lambda A + (1-\lambda)g) = \lambda c_V(A) + (1-\lambda)g$, hence $c(\lambda A + (1-\lambda)g) = \lambda c(A) + (1-\lambda)g$. In particular, *C-Independence* holds.

Next, let $h \in c(A)$, $f, p \in A$, then either $U(h) \geq U(f), U(p)$ and hence, $U(h) \geq \lambda U(f) + (1-\lambda)U(p) = U(\lambda f + (1-\lambda)p)$, $h \in c_U(A \cup \{\lambda f + (1-\lambda)p\})$, or $V(h) \geq V(f), V(p)$ and hence, $V(h) \geq \lambda V(f) + (1-\lambda)V(p) = V(\lambda f + (1-\lambda)p)$, $h \in c_V(A \cup \{\lambda f + (1-\lambda)p\})$, or both. Therefore, $h \in c(A \cup \{\lambda f + (1-\lambda)p\})$, proving *No-C-Hedging*.

Assume $g(s) \notin c(\{f(s), g(s)\})$ $\forall s \in S$. Then $g(s)(x) < f(s)(x)$ and $g(s)(y) < f(s)(y)$ $\forall s \in S$, hence $U(g) < U(f)$ and $V(g) < V(f)$, implying $g \notin c(\{f, g\})$. *Strict Monotonicity* holds.

Let $h \notin A \cup \{f\}$, $h \notin A \cup \{g\}$. If $U(h) \geq U(h')$ $\forall h' \in A$, then $U(h) < U(f), U(g)$, and $U(h) < \lambda U(f) +$

$(1-\lambda)U(g) = U(\lambda f + (1-\lambda)g)$; it follows that in any case, $h \notin c_U(A \cup \{\lambda f + (1-\lambda)g\})$. Similarly, if $V(h) \geq V(h')\ \forall h' \in A$, then $V(h) < V(f), V(g)$, and $V(h) < \lambda V(f) + (1-\lambda)V(g) = V(\lambda f + (1-\lambda)g)$, and in any case, $h \notin c_V(A \cup \{\lambda f + (1-\lambda)g\})$. Therefore, $h \notin c(A \cup \{\lambda f + (1-\lambda)g\})$. *Indirect Ambiguity Aversion* holds.

Since U, V are continuous, $\{(f,A) | f \in c_U(A)\}$ and $\{(f,A) | f \in c_V(A)\}$ are closed. Hence, $\{(f,A) | f \in c(A)\} = \{(f,A) | f \in c_U(A)\} \cup \{(f,A) | f \in c_V(A)\}$ is closed. *Continuity* holds.

Finally, $z \notin c(\{0.5x + 0.5y, z\})$, thus *C-Non-Degeneracy* holds. □

Independence of *C-Independence*. Recall the choice correspondence used in the proof:

$$c(A) \ = \ \arg\max_{f \in A} W(f), \qquad W(f) \ \equiv \ \frac{f(1)(x) + f(2)(x)}{1 + f(2)(x)} = 1 - \frac{1 - f(1)(x)}{1 + f(2)(x)}$$

Since $c(\cdot)$ has continuous utility representation, *WARP*, and, hence, *Framed Ambiguity* holds, and also *Continuity* holds.

Next, note that for all numbers a, b, c, d such that $b, d > 0$ and $a, c \geq 0$, and $\lambda \in [0,1]$,

$$\frac{a}{b} \leq \frac{c}{d} \quad \Longrightarrow \quad \frac{a}{b} \ \leq \ \frac{\lambda a + (1-\lambda)c}{\lambda b + (1-\lambda)d} \ \leq \ \frac{c}{d} \tag{C.2}$$

With $a = f(1)(x) + f(2)(x)$, $b = 1 + f(2)(x)$, $c = g(1)(x) + g(2)(x)$, $d = 1 + g(2)(x)$ for the case when $W(f) \geq W(g)$, and similar substitution for the other case, we get

$$\min\{W(f), W(g)\} \leq W(\lambda f + (1-\lambda)g) \leq \max\{W(f), W(g)\}$$

If $h \in c(A)$, $f, p \in A$, then $W(\lambda f + (1-\lambda)g) \leq \max\{W(f), W(g)\} \leq W(h)$, hence $h \in c(A \cup \{\lambda f + (1-\lambda)p\})$, proving *No-C-Hedging*.

Note that $W(p) \geq W(q)$ if and only if $p(x) \geq q(x)$. Hence, if $g(s) \notin c(\{f(s), g(s)\})\ \forall s \in S$, then $f(1)(x) > g(1)(x)$ and $f(2)(x) > g(2)(x)$. Since $W(\cdot)$ is strictly monotone with respect to $f(1)(x)$ and $f(2)(x)$, $W(f) > W(g)$, hence $g \notin c(\{f, g\})$. *Strict Monotonicity* holds.

Let $h \in A$, $h \notin A \cup \{f\}$, $h \notin A \cup \{g\}$. Then either $W(h') > W(h)$ for $h' \in A$, in which case $h \in c(A \cup \{\lambda f + (1-\lambda)p\})$, or $W(f) > W(h)$ and $W(g) > W(h)$. In this case, $W(h) < \min\{W(f), W(g)\} \leq W(\lambda f + (1-\lambda)g)$, and again, $h \in c(A \cup \{\lambda f + (1-\lambda)p\})$. Thus, *Indirect Ambiguity Aversion* holds.

Finally, since $W(y) = 0 < 1 = W(x)$, *C-Non-Degeneracy* holds. □

Independence of *No-C-Hedging*. Recall choice correspondence $c(\cdot)$ used in the proof. Let $x \in X$, $0 < \underline{\mu} \leq \overline{\mu} \leq 1$. Then

$$c(B) = \bigcup_{k \in [0,\infty)} \arg\max_{f \in B} W_k(f),$$

where

$$W_k(f) = \min\left\{(1-\overline{\mu})f(1)(x) + \overline{\mu}f(2)(x), (1-\underline{\mu})f(1)(x) + \underline{\mu}f(2)(x), f(1)(x) + \underline{\mu}k\right\}$$

Let $f \in c(A\cup B)\cap A$, then $\exists k \geq 0$: $W_k(f) \geq W_k(g)$ $\forall g \in A \subseteq A\cup B$, hence $f \in c(A)$. Thus, α holds.

Let $f \notin c(A\cup\{f\})$. Consider arbitrary $h \in c(f)$, then $\exists k \geq 0$: $W_k(f) \geq W_k(g)$ $\forall g \in A$. If $W_k(f) > W_k(h)$, then $f \in c(A\cup\{f\})$, in contradiction. Hence, $W_k(h) \geq W_k(g)$ $\forall g \in A\cup\{f\}$. It follows $c(A) \subseteq c(A\cup\{f\})$. Hence, *Aizerman's Property* holds.

Note that $W(p) = p(x)$ for all $p \in H_0$. Therefore, *C-β* holds. Also, $y \notin c(\{x,y\})$ for $y \in X\setminus\{x\}$, proving *C-Non-Degeneracy*.

Note that $W_k(\lambda f + (1-\lambda)p) = \lambda W_{k/\lambda}(f) + (1-\lambda)p(x)$ for all $p \in H_0$, $f \in H$, $k \geq 0$, $\lambda > 0$. Thus, $W_k(\lambda f + (1-\lambda)p) \geq W_k(\lambda g + (1-\lambda)p) \iff \lambda W_{k/\lambda}(f) + (1-\lambda)p(x) \geq \lambda W_{k/\lambda}(g) + (1-\lambda)p(x) \iff W_{k/\lambda}(f) \geq W_{k/\lambda}(g)$. To see that *C-Independence* holds, note that, using $k' = k/\lambda$, we get

$$\exists k \geq 0 : W_k(\lambda f + (1-\lambda)p) \geq W_k(\lambda g + (1-\lambda)p) \forall g \in A \iff \exists k' \geq 0 : W_{k'}(f) \geq W_{k'}(g) \forall g \in A$$

Assume $g(s) \notin c(\{f(s), g(s)\})$ $\forall s \in S$. Then, $g(1)(x) < f(1)(x)$ and $g(2)(x) < f(2)(x)$. It follows $W_k(g) < W_k(f)$ for all $k \geq 0$, hence, $g \notin c(\{f,g\})$. *Strict Monotonicity* holds.

Next, let $h \in A$, $h \notin A\cup\{f\}$, $h \notin A\cup\{g\}$. Note that

$$\min\{W_k(f), W_k(g)\} = \min\Big\{\min\left\{(1-\overline{\mu})f(1)(x) + \overline{\mu}f(2)(x), (1-\underline{\mu})f(1)(x) + \underline{\mu}f(2)(x), f(1)(x) + \underline{\mu}k\right\},$$
$$\min\left\{(1-\overline{\mu})g(1)(x) + \overline{\mu}g(2)(x), (1-\underline{\mu})g(1)(x) + \underline{\mu}g(2)(x), g(1)(x) + \underline{\mu}k\right\}\Big\} =$$
$$= \min\Big\{\min\left\{(1-\overline{\mu})f(1)(x) + \overline{\mu}f(2)(x), (1-\overline{\mu})g(1)(x) + \overline{\mu}g(2)(x)\right\},$$
$$\min\left\{(1-\underline{\mu})f(1)(x) + \underline{\mu}f(2)(x), (1-\underline{\mu})g(1)(x) + \underline{\mu}g(2)(x)\right\}, \min\left\{f(1)(x) + \underline{\mu}k, g(1)(x) + \underline{\mu}k\right\}\Big\} \leq$$
$$\leq \min\Big\{(1-\overline{\mu})(\lambda f(1)(x) + (1-\lambda)g(1)(x)) + \overline{\mu}(\lambda f(2)(x) + (1-\lambda)g(2)(x)),$$
$$(1-\underline{\mu})(\lambda f(1)(x) + (1-\lambda)g(1)(x) + \underline{\mu}(\lambda f(2)(x) + (1-\lambda)g(2)(x)), \lambda f(1)(x) + (1-\lambda)g(1)(x) + \underline{\mu}k\Big\} =$$
$$= W_k(\lambda f + (1-\lambda)g)$$

let $\mathbb{K} = \{k \geq 0 | W_k(h) \geq W_k(h') \forall h' \in A\}$, then $W_k(h) < \min\{W_k(f), W_k(g)\} \leq W_k(\lambda f + (1-\lambda)g)$ for all $k \in \mathbb{K}$. Thus, for all $k \geq 0$, $\exists h' \in A\cup\{\lambda f + (1-\lambda)p\} : W_k(h') > W_k(h)$. Hence, $h \notin c(A\cup\{\lambda f + (1-\lambda)p\})$, proving *Indirect Ambiguity Aversion.*

Let $(A^l, f^l) \longrightarrow (A, f) : f^l \in c(A^l)\ \forall l = 1,2,...$ Then $f^l \in A^l$, and $\rho(\{f\}, A) \le d(f, f^l) + \rho(\{f^l\}, A) \le d(f, f^l) + \rho(A^l, A) \longrightarrow 0$, hence $f \in cl(A) = A$. Note that $W_k(f) = W_1(f)$ for all $k > 1$. Therefore,

$$c(B) = \bigcup_{k\in[0,\infty)} \arg\max_{f\in B} W_k(f) = \bigcup_{k\in[0,1]} \arg\max_{f\in B} W_k(f)$$

Since $f^l \in c(A^l)$ for $l = 1,2,...$, $\exists k^l \in [0,1]$: $W_{k^l}(f^l) \ge W_{k^l}(h)\ \forall h \in A^l$. Since $[0,1]$ is compact, $\exists k^{l_m} \longrightarrow k \in [0,1]$. Consider arbitrary $g \in A$; since $A^{l_m} \longrightarrow A$, $\exists g^{l_m} \in A^{l_m}$: $g^{l_m} \longrightarrow g$. Then $W_{p^{l_m}}(f^{l_m}) \ge W_{p^{l_m}}(g^{l_m})\ \forall m$. Note that $(j, h, h') \to W_j(h) - W_j(h')$ is continuous, hence $W_k(f) \ge W_k(g)$. Therefore, $f \in c(A)$. *Continuity* holds. □

Independence of *Strict Monotonocity*. Recall choice correspondence $c(A) = \arg\max_{f\in A} W(f)$, $W(f) = 2f(1)(x) - f(2)(x)$ used in the proof.

Since $c(\cdot)$ has continuous utility representation, *WARP*, and, hence, *Framed Ambiguity* holds, and also *Continuity* holds. Next, since $W(\lambda f + (1-\lambda)g) = \lambda W(f) + (1-\lambda)W(g)$, *C-Independence*, *No-C-Hedging*, and *Indirect Ambiguity Aversion* hold. Finally, $W(x) = 1$, $W(y) = 0$ for $y \in X\setminus\{x\}$, hence $y \notin c(\{x, y\})$, and *C-Non-Degeneracy* holds. □

Independence of Indirect Ambiguity Aversion. Recall choice correspondence $c(A) = \arg\max_{f\in A} W(f)$ with $W(f) = \max\{f(1)(x), f(2)(x)\}$ used in the proof.

Since $c(\cdot)$ has continuous utility representation, *WARP*, and, hence, *Framed Ambiguity* holds, and also *Continuity* holds. Since $W(\lambda f + (1-\lambda)p) = \lambda W(f) + (1-\lambda)p(x)$ for all $f \in H$ $p \in H_0$, *C-Independence* and *No-C-Hedging* hold. Next, if $g(s) \notin c(\{f(s), g(s)\})\ \forall s \in S$, then $f(1)(x) > g(1)(x)$ and $f(2)(x) > g(2)(x)$, hence $W(f) > W(g)$. Therefore, *Strict Monotonicity* holds. Finally, $W(x) = 1 > 0 = W(y)$ for $y \in X\setminus\{x\}$, hence *C-Non-Degeneracy* holds. □

Independence of *Continuity*. Recall choice correspondence $c(A) = \{f \in A | f \succeq g\ \forall g \in A\}$ with $f \succeq g$ if either $f(1)(x) > g(1)(x)$, or $[f(1)(x) = g(1)(x)$ and $f(2)(x) \ge g(2)(x)]$ used in the proof.

Since $\succeq$ is complete and transitive, $c(\cdot)$ satisfies *WARP* and, hence *Framed Uncertainty*. Since for any $\lambda \in (0,1)$, any $f, g, h \in H$ we have $f \succeq g \iff \lambda f + (1-\lambda)h \succeq \lambda g + (1-\lambda)h$, C-Independece, *No-C-Hedging*, and *Indirect Ambiguity Aversion* hold for $c(\cdot)$. Next, $g(s) \notin c(\{f(s), g(s)\})$ implies $g(1) \prec f(1)$, hence $g(1)(x) < f(1)(x)$, $g \prec f$, and $g \notin c(\{f, g\})$, proving *Strict Monotonicity*. Finally, $x \succ y$ for $y \in X\setminus\{x\}$, which yields *C-Non-Degeneracy* for $c(\cdot)$.

Independence of *C-Non-Degeneracy*. Choice correspondence $c(A) = A$ considered in the proof satisfies *Framed Uncertainty*, *C-Independence*, *No-C-Hedging* because the conclusions of these axioms are always satisfied for $c(\cdot)$ independent of the premise. Similarly, $c(\cdot)$ satisfies *Strict Monotonicity* and *Indirect Ambiguity Aversion* because the premises of these axioms are never satisfied. Finally, $\{(A,f)\in\mathcal{K}\times H | f\in c(A)\}=\mathcal{K}\times H$ is closed, hence $c(\cdot)$ is continuous. □

D. Omitted Proofs for Lemma 27.

Recall choice correspondence $c(A)=\left\{f\in A \middle| f(1)(x)\geq f(2)(x) \text{ or } f(2)(x)\geq g(2)(x)\ \forall g\in A\right\}$ considered in Lemma 27. In the paper, we show that $c(\cdot)$ satisfies $c(A)\neq A$ for some $A\in\mathcal{K}$, $c(\{f,g\})\neq\{f,g\}$ for some $f,g\in H$, $c(A)=A$ for all $A\in\mathcal{K}_0$ (and in particular, $c(\cdot)$ violates *C-Non-Degeneracy*). Here, we show that $c(\cdot)$ satisfies Axioms 1–6.

Let $f\in c(A\cup B)\cap A$. If $f(1)\geq f(2)$, then $f\in c(A)$. If $f(1)<f(2)$, then $f(2)\geq g(2)$ for all $g\in A\subseteq A\cup B$, thus $f\in c(A)$. Hence, α holds.

Let $h\in c(A)$. If $h(1)\geq h(2)$, then $h\in c(A\cup\{f\})$. Otherwise $h(2)\geq g(2)$ for all $g\in A$. Then if $h(2)\geq f(2)$, then $h\in c(A\cup\{f\})$. Otherwise $f(2)>h(2)\geq g(2)$ for all $g\in A$, hence $f\in c(A\cup\{f\})$, in contradiction. Therefore, *Aizerman's Property* holds.

Since $c(A)=A$ for all $A\in\mathcal{K}_0$, *C-β* holds.

For $p\in H_0$ denote by $f_\lambda=\lambda f+(1-\lambda)p$. Since $f_\lambda(1)\geq f_\lambda(2) \iff f(1)\geq f(2)$ and $[f_\lambda(2)\geq g_\lambda(2)\ \forall g_\lambda\in\lambda A+(1-\lambda)p]\iff[f_\lambda(2)\geq g_\lambda(2)\ \forall g\in A]$, *C-Independence* holds.

Let $h\in c(A)$ with $f,p\in A$. If $h(1)(x)\geq h(2)(x)$, then $h\in c(A\cup\{\lambda f+(1-\lambda)p\})$. Otherwise, $h(2)(x)\geq g(2)(x)$ for all $g\in A$, and in particular, $h(2)(x)\geq f(2)(x),p(x)$. Therefore, $h(2)(x)\geq \lambda f(2)(x)+(1-\lambda)p(2)(x)$, and hence, $h(2)(x)\geq g(2)(x)$ for all $g\in A\cup\{\lambda f+(1-\lambda)p\}$, and $h\in c(A\cup\{\lambda f+(1-\lambda)p\})$, proving *No-C-Hedging*.

Since $c(\{f(s),g(s)\})=\{f(s),g(s)\}$ for any $f,g\in H$, $s\in S$, *Strict Monotonicity* holds[1].

Consider $h\in A$, $h\notin c(A\cup\{f\})$, $h\notin c(A\cup\{g\})$. If $h\notin c(A)$, by the proven condition α, we have $h\notin c(A\cup\{\lambda f+(1-\lambda)g\})$. Let $h\in c(A)$. If $h(1)(x)\geq h(2)(x)$, then $h\in c(A\cup\{f\})$, in contradiction. Hence, $h(1)(x)<h(2)(x)$, and $h(2)(x)<f(2)(x),g(2)(x)$. Thus, $h(2)(x)<\lambda f(2)(x)+(1-\lambda)g(2)(x)$. Since also $h(1)(x)<h(2)(x)$, then $h\notin c(A\cup\{\lambda f+(1-\lambda)g\})$, proving *Indirect Ambiguity Aversion*.

Let $(A^n,f^n)\to(A,f)$ be such that $f^n\in c(A^n)$ for all $n=1,2,\dots$. If $f(1)(x)\geq f(2)(x)$, then $f\in c(A)$. Otherwise, $f(1)(x)<f(2)(x)$ and $f^n(1)(x)<f^n(2)(x)$ for large enough n; WLOG this is true all n. Therefore, $f^n(2)(x)\geq g^n(2)(x)$ for all $g^n\in A^n$. Consider arbitrary $g\in A$, then $\exists g^n\to g$: $g^n\in A^n$

[1]But its weak counterpart, axiom *Monotonicity* defined in the proof of Theorem 1 in the paper, fails.

for all n. Note that the function $u: \mathcal{K} \times H \to R$ given by $u(A,h) = h(2)(x) - \max_{g \in A} g(2)(x)$ is continuous; therefore, since $f^n(2)(x) - \max_{g \in A^n} g(2)(x) \geq 0$ for all n, then $f(2)(x) - \max_{g \in A} g(2)(x) \geq 0$. Hence, $f \in c(A)$, proving *Continuity*.

Notice that function $v: \mathcal{K} \to R$ given by $v(A) = \max_{g \in A} g(2)$ is continuous, thus function $u: \mathcal{K} \times H \to R$ given by $u(h,A) = h(2) - \max_{g \in A} g(2)$ is continuous. Hence, $f^n(2) - \max_{g \in A^n} g(2) = 0$ for all large enough n implies $f(2) - \max_{g \in A} g(2) = 0$ in the limit. It follows that $f(2) \geq g(2)$ for all $g \in A$, hence $f \in c(A)$, proving *Continuity*. ■